

# Canonical (anti-)commutation rules in QCD and unbroken gauge invariance

## QCD - the two central local anomalies and canonical structure

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### Abstract

The regularities at large distances of complete gauge invariance in QCD are shown to bear nontrivial consequences for the selection among inequivalent representations of canonical commutation (anticommutation) rules for gauge boson (quark) fields.

The trace anomaly forces a modification of the gauge boson Lagrangean and by this of the entire associated canonical structure .

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## 1 - Introduction

Let me limit this introductory section to characterizations of the topics presented .

These topics form three parts :

First part

section 2 - How unique is an axial temporal gauge ?  
 subsections 2a , 2b , 2c  
 first part of section 3 Canonical quantization in an axial gauge  
 subsections 3a , 3 A

Second and main part, devoted to the essential modificatins of the trace anomaly restricted to QCD , beyond the leading operator(s) in the perturbative , i.e. asymptotically free environment

central part of section 3  
 subsections 3I , 3C , 3C1 , 3CD , 3D

Third part , remaining subsections of section 3

subsection 3b , 3c

## 2 - How unique is a temporal axial gauge ?

We begin studying the problem of parallel transport and the equations determining a gauge in which the time component of the connection vanishes, for complete connections with respect to a simple local compact structure group  $\mathcal{G}$

$$\begin{aligned} (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) &= \mathcal{W}_\mu^r(x)(d_r)_{\alpha\beta} \\ d_r &= -d_r^\dagger = \frac{1}{i} J_r \in Lie(\mathcal{D}) ; [d_p, d_q] = f_{pqr} d_r \\ r, p, q &= 1, \dots, \dim \mathcal{G} ; \alpha, \beta = 1, \dots, \dim \mathcal{D} \end{aligned} \quad (1)$$

The quantities defined in eq. 1 follow the notation in the notefile [1-2011], recapitulated below

$$\begin{aligned} (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) &: \text{local operator } \times \mathcal{D} - \text{representation valued} \\ &\quad \text{connection over flat space time } x \\ d_r \in Lie(\mathcal{D}) &: \text{basis of antihermitian matrices forming} \\ &\quad \text{an irreducible representation of the} \\ &\quad \text{Lie algebra of } \mathcal{G} \\ \mathcal{W}_\mu^r(x) &: 4 \times \dim \mathcal{G} \text{ components of hermitian local} \\ &\quad \text{connection fields} \end{aligned} \quad (2)$$

A connection is called complete, if all regularity- , differentiability- and integrability conditions extend to the complete ring of representations  $\mathcal{R} |_{\mathcal{G}}$  , pertaining to  $\mathcal{G}$  [2-1968, 1-2011] .

The notion of complete connection is straightforwardly defined for classical connection-field configurations, whence entering a path integral description, but much less so directly for  $4 \times \dim \mathcal{G}$  components of hermitian local connection fields, as defined in eq. 2 . At this stage we do not attempt to substitute a precise definition.

Instead we look for an operator valued assembly of local gauge transformations denoted  $\mathcal{A} \leftarrow \Omega_{general}$  which achieve an axial gauge condition

$$\left\{ \begin{aligned} & \left( \mathcal{A}(\mathcal{D}) \right)_{\alpha\beta}(x) \end{aligned} \right\} \text{ with } n^\mu \mathcal{W}_\mu(\mathcal{D})\mathcal{A} = 0 \\ \left\{ \mathcal{W}_\mu(\mathcal{D})\mathcal{A} = \mathcal{A}\mathcal{W}_\mu(\mathcal{D})\mathcal{A}^{-1} + \mathcal{A}\partial_\mu\mathcal{A}^{-1} \right\}(x) \quad (3) \\ \text{e.g. : } n^\mu = (1, \vec{0}) ; \text{ independent of } x$$

Eq. 3 takes the form of a first order differential equation for the quantity  $\mathcal{A}' = \mathcal{A}^{-1}(x)$

$$\begin{aligned} -n^\mu \partial_\mu \mathcal{A}'(x) &= n^\mu \mathcal{W}_\mu(x) \mathcal{A}'(x) \Big|_{\mathcal{D}} ; \mathcal{A}' = \mathcal{A}^{-1}(x) \\ \text{with the unitarity condition } &(\mathcal{A}')^\dagger \mathcal{A}' = \mathbb{1}_{\dim \mathcal{D} \times \dim \mathcal{D}} \end{aligned} \quad (4)$$

assuming the connection component  $n^\mu \mathcal{W}_\mu(x) = w(x)$  as given  $\forall x$ .

Adapting the four vector  $n^\mu$  to the special form in eq. 3 , treating the spacelike part of  $x = (t, \vec{x})$  as a parameter and substituting  $U \equiv \mathcal{A}' = \mathcal{A}^{-1}$ , eq. 4 takes the form

$$\begin{aligned} \frac{d}{dt} U(t) \Big|_{\vec{x}} &= -w(t) \Big|_{\vec{x}} U \Big|_{\vec{x}} \rightarrow \dot{\phantom{U}} = \frac{d}{dt} \\ \dot{U}(t) &= -w(t) U(t) ; U^\dagger U = \mathbb{1}_{\dim \mathcal{D} \times \dim \mathcal{D}} \\ U &= \left( \Omega' \right)^{-1}(x) \end{aligned} \quad (5)$$

## 2 a - Integration of eq. 5 , interval by interval

The matrix valued operator extension implicit in eq. 5 poses no problem to its integration , on any forward and/or backward time interval  $I(\tau_1, \tau_0) : \{t | \tau_1 \geq t \geq \tau_0\}$  , the backward time intervals resulting from a time reversal operation. To see this we rewrite eq. 5

$$\begin{aligned} \dot{U}(t) &= -w(t) U(t) ; \quad w_{\alpha\beta}^\dagger = W_0^r(\vec{d}_r)_{\beta\alpha} = -w_{\alpha\beta} \\ & \quad (W_\mu^r)^{op.\dagger} = W_\mu^r \end{aligned} \quad (6)$$

The suffix  $^{op.\dagger}$  in eq. 6 characterizes the local connection fields  $W_\mu^r$  as beeing real, i.e. hermitian or selfadjoint local fields , here with incomplete

association with actual selfadjoint operators in a – connection extended – Hilbert space.

We endow the differential equation in eq. 6 with the interval associated initial condition

$$\begin{aligned} U(t) &\rightarrow U(\tau_1, \tau_0; t) \\ \dot{U}(t) &= -w(t)U(t); U(t = \tau_0) = \mathbb{1}, t \in I(\tau_1, \tau_0) \end{aligned} \quad (7)$$

In eq. 7 –  $\mathbb{1}$  – stands for the unit operator in the direct product  $Lie \mathcal{D} \times \mathcal{H}(x)$ , where  $\mathcal{H}(x)$  shall represent the Hilbert space dissociated from color, spanned by all gauge invariant (local) fields generated from the (complete) connections and color carrying matter fields (quark and antiquark flavors).

The structure of eq. 7 but stripped of the Lie algebra structure is well known from the differential equation for the time evolution operator

$$\begin{aligned} U(t) &\rightarrow E(t) = \exp(iH_0 t) \exp(-iH t) \\ H &= H_0 + H_I : \begin{array}{l} \text{conserved} \\ \text{Hamilton operator} \end{array} ; \begin{array}{l} H, H_0, H_I \\ \text{independent of time} \end{array} \\ \dot{E}(t) &= -\{iH_I(t)\} E(t) \\ H_I(t) &= \exp(iH_0 t) H_I \exp(-iH_0 t) \end{aligned} \quad (8)$$

Despite this similarity, the charge-like gauge structure inherent to eq. 7 develops its own subtleties. The solution is obtained by iteration and is represented by the infinite sum

$$\begin{aligned} U(t - \tau_0) &= \mathbb{1} + \\ &+ \sum_{n=1}^{\infty} (-1)^n \int_0^{\Delta} dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \times \\ &\times w(t_n) w(t_{n-1}) \cdots w(t_1) \\ \Delta &= \tau_1 - \tau_0 \rightarrow \text{for } \Delta > 0 : \tau_n \geq \tau_{n-1} \geq \cdots \geq t_1 \end{aligned} \quad (9)$$

As distinguished before, in the restricted case, where  $w(t, \vec{x})$  are given classical field configurations the convergence and range of the matrix valued functions  $U(\tau_1, \tau_0; t)$  resulting from the form of the solution to eq. 7 presents conditions on admitted given classical field configurations within complete connections  $w(t, \vec{x})$ .

The interval associated solutions to eq. 7,  $U(\tau_1, \tau_0; t)$ , as defined in eq. 9 are best subsummed by path ordered specified integrals of the connection 1-forms – *not* satisfying any gauge fixing conditions –

$$(\mathcal{W}^{(1)}(\mathcal{D}))_{\alpha\beta} = (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) dx^\mu \longrightarrow \mathcal{W}^{(1)}(\mathcal{D}) \quad (10)$$

over oriented (straight) lines associated to these intervals

$$C = C \{ \bar{x} \} = \{ \bar{x} \mid \bar{x}^\mu(\tau) = x_0^\mu + (x_1 - x_0)^\mu \tau \} \quad (11)$$

$$0 \leq \tau \leq 1$$

$$U \left( x \stackrel{C}{\leftarrow} y \right) = P \exp \left( - \int_C W^{(1)}(\bar{x}) \right)$$

$$\rightarrow (U(x, C, y))_{\alpha\beta} \in \mathcal{D}(\mathcal{G})$$


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$$C = C \{ \bar{x} \} :$$

$$\bar{x}(\tau) ; 1 \geq \tau \geq 0 ; \tau : \text{path parameter}$$

$$\bar{x}(\tau = 1) = x \leftarrow x_1 ; \bar{x}(\tau = 0) = y \leftarrow x_0 \quad (12)$$

In eq. 12  $P$  denotes matrix as well as operator ordering along the path  $C$  since  $W^{(1)}$  is matrix and operator valued.

Since we want to keep differentiability criteria, we abstain for the moment from considering polygon like convolutions, of – a finite number – of segmentwise straight lines, but allow only continuation of segments along one straight line to arbitrary but finite extension. This can be done maintaining the parameter segment  $0 \leq \tau \leq 1$  for individual segments, assuming that the series expansion of the quantities  $U \left( x_r \stackrel{C_r}{\leftarrow} y_r \right) ; r = 1, 2, \dots, M < \infty$  of the form given in eq. 12 are well defined for individual parallel segments along one straight line – in configuration space.

The convolution of 2 so restricted segments continuing each other then takes the form

$$U \left( x_2 \stackrel{C_2}{\leftarrow} y_2 = x_1 \right) U \left( x_1 \stackrel{C_1}{\leftarrow} y_1 \right) = U \left( x_2 \stackrel{C_{2\&1}}{\leftarrow} y_1 \right)$$

$$C_1 = \{ \bar{x}(\tau) = y_1 + (x_1 - y_1)\tau \}$$

$$C_2 = \{ \bar{x}(\tau) = x_1 + (x_2 - x_1)\tau \} ; 0 \leq \tau \leq 1$$

$$C_{2\&1} = \{ \bar{x}(\tau) = y_1 + (x_2 - y_1)\tau \}$$

$$\text{with } : x_1 - y_1 // x_2 - x_1 // x_2 - y_1 \quad (13)$$

The family of unitary transformations, solutions to eq. 7, corresponds to the choice of segments along the straight line with fixed  $\vec{x}$  coordinates

$$U(\tau_1, \tau_0; \Delta t) = U \left( x \stackrel{C}{\leftarrow} y \right) ;$$

$$x = (\tau_0 + \Delta t, \vec{x}) ; \Delta t = t - \tau_0 \quad (14)$$

$$y = (\tau_0, \vec{x}) ; \tau_1 \geq t \geq \tau_0$$

## 2 b - Reparametrization of straight line using a tangent vector

$$\underline{v} = ( \underline{v}^\mu )$$

We want to use a variant of the parametrizations of the parallel transport operators  $U \left( x \stackrel{\mathcal{C}}{\leftarrow} y \right) = P \exp \left( - \int_C W^{(1)} ( \bar{x} ) \right)$  introduced in eq. 12 , following ref. [1-2011] ( eqs. 77-79, op.cit. )

$$\begin{aligned} U \left( x \stackrel{\mathcal{C}}{\leftarrow} y \right) &= P \exp \left( - \int_C W^{(1)} ( \bar{x} ) \right) \rightarrow \\ &\rightarrow ( U ( x , C , y ) )_{\alpha\beta} \in \mathcal{D} ( \mathcal{G} ) \\ C &= C \{ \bar{x} \} : \bar{x} = \bar{x} ( s ) ; \tau \geq s \geq 0 ; s : \text{path parameter} \\ &\bar{x} ( s = \tau ) = x \\ \bar{x} ( s = 0 ) &= y \end{aligned} \tag{15}$$

Next we use the explicit parametric dependence  $x = x ( \tau )$  and assume that there exists a continous extension of the functional dependence as defined in eq. 15 into a small neighbourhood in all of space-time around the point  $x ( \tau )$  defined e.g. with a Lorentz noninvariant Euclidean norm  $\| \xi \|^2 = ( \xi^0 )^2 + ( \vec{\xi} )^2$

$$U \left( x ( \tau ) \stackrel{\mathcal{C}}{\leftarrow} y \right) \rightarrow \tilde{U} ( x ( \tau ) + \xi ) ; \| \xi \| < \delta \tag{16}$$

The differential equation ( eqs. 5 , 6 ) takes the form

$$\begin{aligned} \frac{d}{d\tau} U \left( x ( \tau ) \stackrel{\mathcal{C}}{\leftarrow} y \right) &= - \underline{v}^\mu \mathcal{W}_\mu ( x ( \tau ) ) U \left( x ( \tau ) \stackrel{\mathcal{C}}{\leftarrow} y \right) \\ \frac{d}{d\tau} \tilde{U} \left( x ( \tau ) \stackrel{\mathcal{C}}{\leftarrow} y \right) &= \underline{v}^\mu \partial_{\xi^\mu} \tilde{U} ( x ( \tau ) + \xi ) \Big|_{\xi=0} \rightarrow \\ \rightarrow \underline{v}^\mu ( \partial_{x^\mu} + \mathcal{W}_\mu ( x ) ) \tilde{U} ( x ) \Big|_{x=x(\tau)} &= 0 \\ \tilde{U} &= \tilde{U} ( x ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y ) ; \\ \tilde{U} ( y ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y ) &= \P \end{aligned} \tag{17}$$

The notation of the arguments  $x ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y$  of the parallel transport operators  $\tilde{U}$  in the last line of eq. 17 shall indicate that x is understood as space time variable, whereas the entire connection  $\{ \mathcal{W}^{(1)} ( \mathcal{D} ) \}$  as well as the base point y shall be understood in a parametric sense.

The relations in the last two lines of eq. 17 show the intrinsic connection

between boundary - and integrability conditions pertaining to complete connections and the associated determination of an axial gauge.

The 'variable-' point  $x$  ( 'Aufpunkt' in german terminology ) and base point  $y$  represented by the endpoints along the straight line of parallel transport through the quantity

$$\tilde{U} ( x ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y ) \quad (18)$$

separate the adjoint representation of the local limit into the product of two distinct local representations of the gauge group , one each at  $x$  and  $y$  , transforming under relatively complex conjugate but singly arbitrary irreducible representations  $\mathcal{D} \leftrightarrow x$  ,  $\overline{\mathcal{D}} \leftrightarrow y$  of the gauge group  $\mathcal{G}$  – subject to restrictions from continuity of local gauge group representations – for complete connections.

This gives rise to the  $(x, y)$  associated bi-local transformation properties under an – operator valued – local family of gauge transformations discussed in the next subsection below.

**2 c - QCD  $\mathcal{D}(\mathcal{G})$  – strings**  
**building blocks of bi-local gauge covariant parallel transports**  
 $x \xleftarrow{\mathcal{G}} y$  , **relative to the associated irreducible representations**  
 $\mathcal{D} \leftrightarrow x$  ,  $\overline{\mathcal{D}} \leftrightarrow y$  **of the gauge group  $\mathcal{G}$**

In the following we drop the  $\tilde{U} \longrightarrow U$  symbol explained in eqs. 17 - 18 for simplicity and repeat the differential 'parallel-transport'- equation ( eq. 17 )

$$\begin{aligned} \underline{y}^\mu ( \partial_{x^\mu} + \mathcal{W}_\mu ( x ) ) U ( x ) \big|_{x=x(\tau)} &= 0 \\ U ( x ) &\longleftarrow U ( x ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y ) \\ U ( y ; \{ \mathcal{W}^{(1)} ( \mathcal{D} ) \} , y ) &= \P \end{aligned} \quad (19)$$

Next we consider local gauge transformations  $\{ \Omega \}$  , substitute the identity  $\Omega^{-1} ( x ) \Omega ( x ) = \P$  and use the chain rule for partial derivatives

$$\begin{aligned} \underline{y}^\mu ( \partial_{x^\mu} + \mathcal{W}_\mu ( x ) ) ( \Omega^{-1} ( x ) \Omega ( x ) ) U ( x ) \big|_{x=x(\tau)} &= 0 \\ \underline{y}^\mu \Omega^{-1} ( x ) \left( \begin{array}{c} \Omega ( x ) [ \partial_{x^\mu} + \mathcal{W}_\mu ( x ) ] \Omega^{-1} ( x ) \\ + \partial_{x^\mu} \end{array} \right) \Omega ( x ) U ( x ) \bigg|_{x=x(\tau)} & \\ = 0 & \end{aligned} \quad (20)$$

By eq. 3 the quantity in [ ] brackets in the second relation of eq. 20 represents the gauge transformed connection

$$\Omega ( x ) [ \partial_{x^\mu} + \mathcal{W}_\mu ( x ) ] \Omega^{-1} ( x ) = \mathcal{W}_\mu^\Omega ( x ) \quad (21)$$

And so we derive the bi-locally transformed

parallel transport operators, from the associated differential equation and initial conditions ( eqs. 19 , 20 )

$$\begin{aligned}
& \left. \begin{aligned} & \underline{v}^\mu \left( \partial_{x^\mu} + \mathcal{W}_\mu^\Omega(x) \right) U \left( x; \left\{ \left\{ \mathcal{W}^{(1)}(\mathcal{D}) \right\}^\Omega \right\}, y \right) \Big|_{x=x(\tau)} \\ & = 0 \end{aligned} \right\} \rightarrow \\
& U \left( y; \left\{ \left\{ \mathcal{W}^{(1)}(\mathcal{D}) \right\}^\Omega \right\}, y \right) = \mathbb{I} \\
\hline
& U \left( x; \left\{ \left\{ \mathcal{W}^{(1)}(\mathcal{D}) \right\}^\Omega \right\}, y \right) = \\
& = \Omega(x) U \left( x; \left\{ \mathcal{W}^{(1)}(\mathcal{D}) \right\}, y \right) \Omega^{-1}(y) \\
& \quad \forall \text{ irreducible representations } \mathcal{D}(\mathcal{G}) \text{ and } W^{(1)}, \{ \Omega \} \\
& U \left( x; \left\{ \left\{ \mathcal{W}^{(1)}(\mathcal{D}) \right\} \right\}, y \right) = P \exp \left( - \int_C W^{(1)}(x) \right)
\end{aligned} \tag{22}$$

We emphasize that existence and uniqueness of the operator valued families  $U(\cdot)$ , necessary for the derivation of the transformation properties, displayed in eqs. 15 and 22, constitute regularity conditions imposed on complete connections [1-2011]. Connections  $\mathcal{W}^{(1)}$  as well as local gauge transformations  $\{ \Omega \}$  shall denote general quantities, not related to gauge fixing.

### 3 - Canonical quantization in an axial gauge

We postpone all discussion of multiplicative renormalization factors and identify field strengths, imposing the axial gauge condition ( eqs. 3 - 5 ). Gauge fixed quantities are always singled out by a superfix  $^{\mathcal{A}}$  contrasting with connections and field strengths in a general gauge

$$n^\mu (W_\mu)^{\mathcal{A}}(x) = 0 ; n^\mu = (1, \vec{0}) \tag{23}$$

We follow the notation used in ref. [1-2011] ( eqs. (65) - (72), op.cit. ), with the exception to use  $\mathcal{W}^{(n)}(\mathcal{D})$  for the Lie algebra matrix valued n-forms (  $n = 1, 2$  ).

$$\begin{aligned}
& \left( \mathcal{W}^{(1)}(\mathcal{D}) \right)_{\alpha\beta} = W_\mu^r(x) (d_r(\mathcal{D}))_{\alpha\beta} dx^\mu \\
& W_\mu^r(x) : \text{real} ; r = 1, \dots, G = \dim(\mathcal{G})
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \mathcal{W}^{(2)}(\mathcal{D}) \rightarrow \mathcal{W}^{(2)} = \partial \mathcal{W}^{(1)} + \left( \mathcal{W}^{(1)} \right)^2 ; \partial \equiv dx^\mu \partial_{x^\mu} \\
& \left( \mathcal{W}^{(2)} \right)_{\alpha\beta} = \frac{1}{2} W_{\mu\nu}^r (d_r)_{\alpha\beta} dx^\mu \wedge dx^\nu ; d_r \in \text{Lie}(\mathcal{D}) \\
& \rightarrow \mathcal{W}_{\mu\nu}^{(2)} = \partial_\mu \mathcal{W}_\nu^{(1)} - \partial_\nu \mathcal{W}_\mu^{(1)} + \left[ \mathcal{W}_\mu^{(1)}, \mathcal{W}_\nu^{(1)} \right] \\
& W_{\mu\nu}^r = -W_{\nu\mu}^r = \partial_\mu W_\nu^r - \partial_\nu W_\mu^r + f_{rpq} W_\mu^p W_\nu^q
\end{aligned}$$

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$$\mathcal{W}^{(2)}(\mathcal{D}) \equiv \mathcal{B}^{(2)}(\mathcal{D}) ; W_{\mu\nu}^r \equiv B_{\mu\nu}^r \begin{array}{l} \text{field strength components} \\ \text{independent of } \mathcal{D} \end{array} \tag{25}$$



We decompose the field strength vectors and tensors, in axial gauge, with the conventions

$$\begin{aligned}
W_{\mu}^r{}^{\mathcal{A}} &= \eta_{\mu\nu} W^{r\nu\mathcal{A}} ; W^{r0\mathcal{A}} = 0 , W^{rk\mathcal{A}} = \left( \vec{W}^r \right)^k{}^{\mathcal{A}} \\
\eta_{\mu\nu} &= \text{diag} ( 1 , -1 , -1 , -1 ) : \text{flat space metric} ; k = 1, 2, 3 \\
B_{0k}^r{}^{\mathcal{A}} &= - \left( \vec{E}^r \right)^k{}^{\mathcal{A}} ; B_{jl}^r{}^{\mathcal{A}} = \varepsilon_{kjl} \left( \vec{B}^r \right)^k{}^{\mathcal{A}} ; \\
k, j, l &= 1, 2, 3 \\
\vec{w}^r{}^{\mathcal{A}} &\equiv - \vec{W}^r{}^{\mathcal{A}} ; \text{covariantly} : w_{\mu}^r{}^{\mathcal{A}} = - W_{\mu}^r{}^{\mathcal{A}} ; \bullet = \partial_t
\end{aligned} \tag{26}$$

Then the field strengths are expressed through the potentials

$$\begin{aligned}
\vec{E}^r{}^{\mathcal{A}} &= - \vec{\dot{w}}^r{}^{\mathcal{A}} ; \vec{B}^r{}^{\mathcal{A}} = \nabla \wedge \vec{w}^r{}^{\mathcal{A}} + f_{rpq} \vec{w}^{p\mathcal{A}} \wedge \vec{w}^{q\mathcal{A}} \\
\left( \vec{w} \right)^{\mathcal{A}} &= \left( \vec{w}^r \right)^{\mathcal{A}} d_r ; d_r \rightarrow (d_r)_{\alpha\beta}
\end{aligned} \tag{27}$$

It is advantageous to use systematically in parallel the Lie algebra  $\mathcal{D}$  matrix valued quantities completing the one given on the last line of eq. 27

$$\begin{aligned}
\mathcal{A} : \quad \mathcal{B}_{0k}^{(2)} &= - \left( \vec{E} \right)^k ; \quad \vec{E} = \vec{E}^r d_r \\
\mathcal{B}_{jl}^{(2)} &= \varepsilon_{kjl} \left( \vec{B} \right)^k ; \quad \vec{B} = \vec{B}^r d_r
\end{aligned} \tag{28}$$

The axial gauge fields are indicated by  $\mathcal{A}$  : in front of their appearance. Then eq. 27 becomes

$$\mathcal{A} : \quad \vec{E} = - \vec{\dot{w}} ; \quad \vec{B} = \nabla \wedge \vec{w} + \vec{w} \wedge \vec{w} \quad \Big| \quad (\mathcal{D}) \tag{29}$$

In line with the  $\mathcal{D}$  projection we specify the  $\mathcal{D}$  – dependent trace normalization

$$\begin{aligned}
&- \text{tr}_{\mathcal{D}} d^p d^q = \delta^{pq} T_2(\mathcal{D}) \\
&- \sum_r (d^r)^2 = \frac{C_2(\mathcal{D})}{\dim(\mathcal{D})} \P_{\dim \mathcal{D} \times \dim \mathcal{D}} \rightarrow \\
&T_2(\mathcal{D}) = \frac{C_2(\mathcal{D})}{\dim(\mathcal{G})} C_2(\mathcal{D}) ; \text{for } \left. \begin{array}{l} \mathcal{G} = SU3 \\ \mathcal{D} = 3 \text{ or } \bar{3} \end{array} \right\} : \tag{30}
\end{aligned}$$

$$\dim \mathcal{D} = 3 , \dim \mathcal{G} = 8 , T_2(\mathcal{D}) = \frac{1}{2} C_2(\mathcal{D}) = \frac{4}{3}$$

In eq. 30  $C_2(\mathcal{D})$  denotes the eigenvalue of the second order Casimir operator projected on the irreducible representation  $\mathcal{D}$

$$-\sum_r (d^r)^2 = C_2(\mathcal{D}) \mathbb{I}_{\dim \mathcal{D} \times \dim \mathcal{D}} \quad (31)$$

### 3 a - Bare Lagrangean density and equations of motion in unconstrained gauges

The bare local Lagrangean density – postponing the discssion of quark and antiquark (matter-) fields – is formed by the bare field strengths using the relations in eq. 30

$$\begin{aligned} \mathcal{L}(x) &= \frac{1}{4 T_2(\mathcal{D}) g^2} \text{tr}_{\mathcal{D}} [\mathcal{B}_{\mu\nu} \mathcal{B}^{\mu\nu}] (\mathcal{D})(x) + \mathcal{L}_{\{q\}}(x) \\ \mathcal{L}_{\{q\}}(x) &= \sum_{q-fl} \bar{q}^{c'} \left\{ \frac{i}{2} \gamma^\mu \left[ \begin{pmatrix} \vec{D}_\mu(3) \\ - \begin{pmatrix} \overleftarrow{D}_\mu(\bar{3}) \end{pmatrix}^{c'\dot{c}} \end{pmatrix} - m_q \delta_{c'\dot{c}} \right] q^c(x) \right\} \end{aligned} \quad (32)$$

The Euler equations generating an extremum of the bare action take the form

$$\begin{aligned} \partial_\varrho \left\{ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} \right\} - \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} &= 0 \\ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} &= \frac{1}{2 g^2} B^{\nu\mu r} \end{aligned} \quad (33)$$

The extremum is to be taken over variations of the connection components  $\delta W_\sigma^s(x)$  restricted to vanish on a limiting finite or infinite space time domain .

Substitting the variations of the field strengths

$$\begin{aligned} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} &= \delta^{rs} \left( \delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma \right) \\ \frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} &= f_{srt} \left( \delta_\nu^\sigma W_\mu^t - \delta_\mu^\sigma W_\nu^t \right) \end{aligned} \quad (34)$$

the equations of motion become

$$\begin{aligned}
& \partial_\varrho \left\{ \frac{1}{g^2} B^{\sigma\varrho s} \right\} + \frac{1}{g^2} f_{str} W_\varrho^t \{ B^{\sigma\varrho r} \} \\
& (d_t (\mathcal{D} = \text{adjoint representation}))_{sr} = (ad_t)_{sr} = f_{str} \longrightarrow \\
& (D_\varrho (ad))_{sr} \left\{ \frac{1}{g^2} B^{\sigma\varrho r} \right\} = 0 \\
& (D_\varrho (ad))_{sr} = \partial_\varrho \delta_{sr} + W_\varrho^t (ad_t)_{sr}
\end{aligned} \tag{35}$$

Here we repeat the Lagrangean density pertaining to quark-antiquark flavors as defined in eq. 32

$$\begin{aligned}
\mathcal{L}_{\{q\}} &= \\
&= \sum_{q-fl} \bar{q}^{c'} \left\{ \frac{i}{2} \gamma^\mu \left[ \begin{pmatrix} \vec{D}_\mu (3) \\ - \overleftarrow{D}_\mu (\bar{3}) \end{pmatrix}_{c'\dot{c}} \right] - m_q \delta_{c'\dot{c}} \right\} q^c \\
&\begin{pmatrix} \vec{D}_\mu (3) \end{pmatrix}_{c'\dot{c}} = \overrightarrow{\partial}_\mu \delta_{c'\dot{c}} + W_\mu^r \frac{1}{i} \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}} \\
&\begin{pmatrix} \overleftarrow{D}_\mu (\bar{3}) \end{pmatrix}_{\dot{c}c'} = \overleftarrow{\partial}_\mu \delta_{\dot{c}c'} - W_\mu^r \frac{1}{i} \left( \frac{1}{2} \bar{\lambda}^r \right)_{\dot{c}c'} \\
&= \overleftarrow{\partial}_\mu \delta_{c'\dot{c}} - W_\mu^r \frac{1}{i} \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}}
\end{aligned} \tag{36}$$

Substituting the relative to each other complex conjugate representation matrices of  $Lie(SU3_c)$  in eq. 36 it follows

$$\begin{aligned}
\mathcal{L}_{\{q\}} &= \\
&= \sum_{q-fl} \bar{q}^{c'} \left\{ \gamma^\mu \left[ \begin{pmatrix} \frac{i}{2} \overrightarrow{\partial}_\mu \delta_{c'\dot{c}} + \\ + W_\mu^r \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}} \end{pmatrix} \right] - m_q \delta_{c'\dot{c}} \right\} q^c
\end{aligned} \tag{37}$$

In eqs. 36 and 37  $\lambda^r$ ;  $r = 1, \dots, 8$  denote the eight Gell-Mann matrices defining the Lie algebra representation (3) of  $SU3$  and associated structure constants, with the identifications

$$\begin{aligned}
\mathcal{G} &= SU3_c ; \mathcal{D} = 3 : \\
(d_r)_{\alpha\beta} &= \frac{1}{i} \left( \frac{1}{2} \lambda^r \right)_{\alpha\beta} ; \{ \alpha = c', \beta = \dot{c} \} = 1, 2, 3 \\
tr(\mathcal{D} = 3) \left( \frac{1}{2} \lambda^r \right) \left( \frac{1}{2} \lambda^s \right) &= T_2(3) \delta_{rs} ; T_2(3) = \frac{1}{2} \\
\left[ \frac{1}{2} \lambda^r, \frac{1}{2} \lambda^s \right] &= i f_{rst} \frac{1}{2} \lambda^t \\
f_{ruv} f_{suv} &= C_2(ad) \delta_{rs} ; C_2(ad) = 3
\end{aligned} \tag{38}$$

The equations of motion ( eqs. 33 - 35 ) are modified by the quark-antiquark current

$$\begin{aligned} \partial_\varrho \left\{ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} \right\} - \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} &= \frac{\delta \mathcal{L}_{\{q\}}}{\delta W_\sigma^s} \\ \frac{\delta \mathcal{L}_{\{q\}}}{\delta W_\sigma^s} &= \sum_{q-fl} \bar{q}^{\dot{c}'} \left\{ \gamma^\sigma \left( \frac{1}{2} \lambda^s \right)_{c'\dot{c}} \right\} q^c = (j^{\sigma s})_{\{q\}} \end{aligned} \quad (39)$$

Eq. 35 becomes

$$\begin{aligned} (D_\varrho(ad))_{sr} \left\{ \frac{1}{g^2} B^{\sigma\varrho r} \right\} &= (j^{\sigma s})_{\{q\}} \\ (D_\varrho(ad))_{sr} &= \partial_\varrho \delta_{sr} + W_\varrho^t (ad_t)_{sr} ; (ad_t)_{sr} = f_{str} \\ (j^{\sigma s})_{\{q\}} &= \sum_{q-fl} \bar{q}^{\dot{c}'} \left\{ \gamma^\sigma \left( \frac{1}{2} \lambda^s \right)_{c'\dot{c}} \right\} q^c \end{aligned} \quad (40)$$

### 3 A - Canonical commutation rules for gauge fields in axial gauges from bare Lagrangean density and residual fixed time gauge invariance

Because we use the metric  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  as given in eq. 26, choosing as field coordinates the covariant space components  $W_\mu^{tA} \rightarrow W_m^{tA}$ ;  $m = 1, 2, 3$ , some care must be taken with respect to the difference of sign between contra- and covariant space vectors. The above choice corresponds to coordinates  $\hat{q}_m = -\hat{q}^m$ ,  $\hat{p}^n = L_{\partial_t} \hat{q}_n$  for quantum mechanical conjugate variables in uncurved space-time, with the commutation rules

$$[\hat{p}^n, \hat{q}_m] = -[\hat{p}^n, \hat{q}^m] = \frac{1}{i} \delta_m^n \P \quad (41)$$

In order to show things step by step, I repeat two relations from eqs. 33 and 34

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta (\partial_\varrho W_\sigma^s)} &= \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} = \frac{1}{g^2} B^{\sigma\varrho s} \\ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} &= \frac{1}{2g^2} B^{\nu\mu r} \\ \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} &= \delta^{rs} (\delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma) \end{aligned} \quad (42)$$

In the temporal axial gauges we are using it follow from eq. 42

$$\begin{aligned}
\mathcal{A} : \quad \Pi^{s\,m} &= \frac{\delta \mathcal{L}}{\delta (\partial_{\varrho \rightarrow 0} W_{\sigma \rightarrow m}^s)} = \frac{1}{g^2} B^{m0\,s} \\
&= - \frac{1}{g^2} E^{s\,m} \\
\hline
\Pi^{s\,m}(t, \vec{x}) &\sim \hat{p}^{s\,m}(\vec{x}); \quad \left. \begin{aligned} W_n^r(t, \vec{x}) &\sim \hat{q}_n^r(\vec{x}) \\ \hat{q}_n^r(\vec{x}) &\equiv -\hat{q}^{r\,n}(\vec{x}) \end{aligned} \right|_{\substack{\text{gauge} \\ \text{fields}}} \\
- E^{s\,m} &= (W_m^r)^\bullet
\end{aligned} \tag{43}$$

We cast eq. 43 into a 'canonical' form, adapted to a plane with fixed time , whereas  $\vec{x}, \vec{y}, \dots$ , varying on the chosen plane, become continuous labels of canonical variables

$\mathcal{A}, t$	$\Pi^{s\,m}(\vec{x}) = \mathcal{L}, \mathcal{Q}^{\bullet s}_m(\vec{x})$	$\mathcal{Q}_n^r(\vec{y}) = W_n^r(t, \vec{y})$
	$\Pi^{s\,m}(\vec{x}) = \mathcal{L}, \mathcal{Q}^{\bullet s}_m(\vec{x}) = - \frac{1}{g^2} E^{s\,m}(\vec{x}) *$	
	$- E^{s\,m}(\vec{x}) = \mathcal{Q}^{\bullet s}_m(\vec{x}) = W^{\bullet s}_m(t, \vec{x})$	

\* : as stated previously , we do not allow any dependence of the bare coupling constant g on a scale, which enters upon renormalization giving rise to the trace anomaly .

(44)

With canonical variables for gauge fields defined in eq. 44 the associated equal time commutators take the form, for one selected common time  $t$

$$\begin{aligned}
\mathcal{A} : \quad [W_m^r(t, \vec{x}), \Pi^{s\,n}(t, \vec{y})] &= i \delta^{r\,s} \delta_m^n \delta^{(3)}(\vec{x} - \vec{y}) \P \\
[W_m^r(t, \vec{x}), W_n^s(t, \vec{y})] &= 0 \\
[\Pi^{r\,m}(t, \vec{x}), \Pi^{s\,n}(t, \vec{y})] &= 0
\end{aligned} \tag{45}$$

In eq. 45  $\P$  denotes the unit operator in a suitable extension of the Hilbert space of fully gauge invariant states, allowing for the gauge variant canonical variables to be fully represented .

In the follwing we concentrate on the nonvanishing commutation rules of the canonical conjugate variables as displayed in the first relation of eq. 45, which transforms into

$$\begin{aligned}
\mathcal{A} : \quad [W_m^r(t, \vec{x}), E^{s\,n}(t, \vec{y})] &= \frac{1}{i} g^2 \delta^{r\,s} \delta_m^n \delta^{(3)}(\vec{x} - \vec{y}) \P \\
E^{s\,n}(t, \vec{y}) &= - W^{\bullet s}_m(t, \vec{y})
\end{aligned} \tag{46}$$

### 3 A R - Residual fixed time gauge invariance

The conjugate variables in eq. 45 allow further gauge transformations, i.e. are not fixed by the choice

$$\mathcal{A} : W_0^r(t', \vec{x}) = 0 ; \forall t' \quad (47)$$

We repeat the covariant derivative for the adjoint representation eq. 35 for general gauges

$$\begin{aligned} (d_t(\mathcal{D} = \text{adjoint representation}))_{sr} &= (ad_t)_{sr} = f_{str} \\ [ad_p, ad_q] &= f_{pqr} ad_r \\ \text{e.g. : } (D_\varrho(ad))_{sr} \{B_{\sigma\tau}^r\} &= \\ &= \partial_\varrho \{B_{\sigma\tau}^s\} + f_{str} W_\varrho^t \{B_{\sigma\tau}^r\} \end{aligned} \quad (48)$$

We also retrace gauge transformations of the connection fields  $\mathcal{W}_\mu(\mathcal{D})$  relative to the irreducible representation  $\mathcal{D}$  and its Lie algebra  $Lie \mathcal{D}$  as defined in eqs. 1 and 2

$$\begin{aligned} (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) &= \mathcal{W}_\mu^r(x) (d_r)_{\alpha\beta} \\ d_r &= -d_r^\dagger = \frac{1}{i} J_r \in Lie(\mathcal{D}) ; [d_p, d_q] = f_{pqr} d_r \\ r, p, q &= 1, \dots, \dim \mathcal{G} ; \alpha, \beta = 1, \dots, \dim \mathcal{D} \\ (\mathcal{W}_\mu(\mathcal{D}))_{\alpha\beta}(x) &: \begin{array}{l} \text{local operator} \times \\ \mathcal{D} - \text{representation valued} \\ \text{connection over} \\ \text{flat space time } x \end{array} \\ d_r \in Lie(\mathcal{D}) &: \begin{array}{l} \text{basis of antihermitian matrices forming} \\ \text{an irreducible representation of the} \\ \text{Lie algebra of } \mathcal{G} \end{array} \\ \mathcal{W}_\mu^r(x) &: \begin{array}{l} 4 \times \dim \mathcal{G} \text{ components of hermitian} \\ \text{local connection fields} \end{array} \end{aligned} \quad (49)$$

---

\* : Replacing the gauge field Lagrangean density according to the trace anomaly [3-1981]

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{g^2} \mathcal{X} \rightarrow \bar{\mathcal{L}} = -\left[ \frac{1}{\bar{g}^2(\mathcal{X})} - J \right] \mathcal{X} \\ \mathcal{X} &= \frac{1}{4} B_{\mu\nu}^r B^{\mu\nu r} \end{aligned} \quad (50)$$

In eq. 50  $J$  stands for a suitable constant, while  $\bar{g}(\bar{l})$  denotes the scale dependent coupling constant, as discussed also in ref. [4-1978], established

initially in the perturbative domain of QCD

$$\begin{aligned}
\bar{l} &= \log (\bar{\mu} / \mu) = \frac{1}{4} \log (\bar{\mu}^4 / \mu^4) \rightarrow \frac{1}{4} \log (\mathcal{X} / \mu^4) \\
&\rightarrow \frac{1}{8} \log \left( (\mathcal{X} / \mu^4)^2 \right) \\
\bar{g}(\mathcal{X}) &= \bar{g} \left( \bar{l} = \frac{1}{8} \log \left( (\mathcal{X} / \mu^4)^2 \right) \right) \\
\frac{d}{d\bar{l}} \bar{g} &= \beta(\bar{g}) = \bar{g} \bar{\kappa} [ - (b_0 + b_1 \bar{\kappa}^1 + \dots) ] \\
\bar{\kappa} &= \frac{\bar{g}^2}{16 \pi^2}; \bar{\alpha}_s = 4 \pi \bar{\kappa} \\
\frac{d}{d\bar{l}} \bar{g}^{-2} &= \frac{1}{8 \pi^2} b(\bar{\kappa}) \\
b(\bar{\kappa}) &= - (\bar{g} \bar{\kappa})^{-1} \beta(\bar{g}) = (b_0 + b_1 \bar{\kappa}^1 + \dots)
\end{aligned} \tag{51}$$

At this point we recall eq. 34

$$\begin{aligned}
\frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} &= \delta^{rs} (\delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma) \\
\frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} &= f_{srt} (\delta_\nu^\sigma W_\mu^t - \delta_\mu^\sigma W_\nu^t)
\end{aligned} \tag{52}$$

and adapt eq. 42 obtained from eq. 33 to the quantity  $\mathcal{X}$  defined in eq. 50

$$\begin{aligned}
\frac{\delta \mathcal{X}}{\delta (\partial_\varrho W_\sigma^s)} &= \frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} = B^{\varrho\sigma s} \\
\frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} &= \frac{1}{2} B^{\mu\nu r}; \quad \frac{\delta \mathcal{X}}{\delta (W_\sigma^s)} = \frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (W_\sigma^s)} \\
\mathcal{X} &= \frac{1}{4} B_{\mu\nu}^r B^{\mu\nu r} = -g^2 \mathcal{L}_{gauge}
\end{aligned} \tag{53}$$

The variation of the trace anomaly induced Lagrangean density  $\bar{\mathcal{L}}$ , defined

in eq. 50, is obtained by repeated use of the chain rule

$$\begin{array}{l}
\frac{\delta \bar{\mathcal{L}}}{\delta (\partial_\varrho W_\sigma^s)} = \left[ \frac{\delta \bar{\mathcal{L}}}{\delta \mathcal{X}} \frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} \right] \frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} \\
\frac{\delta \bar{\mathcal{L}}}{\delta (W_\sigma^s)} = \left[ \frac{\delta \bar{\mathcal{L}}}{\delta \mathcal{X}} \frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} \right] \frac{\delta B_{\mu\nu}^r}{\delta (W_\sigma^s)} \\
\frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} = \delta^{rs} (\delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma) \\
\frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} = f_{srt} (\delta_\nu^\sigma W_\mu^t - \delta_\mu^\sigma W_\nu^t)
\end{array} \tag{54}$$

We proceed to evaluate the derivatives in eq. 54 using eq. 51 . The quantities in [ . ] brackets in the first two relations of eq. 54 contain expressions wherein both canonically conjugate variables appear. Hence an ordering is necessary. This is straightforward in all such situations and amounts to neglecting all local terms arising from the noncommutativity of the latter.

$$\begin{aligned}
\frac{\delta \bar{\mathcal{L}}}{\delta \mathcal{X}} &= \bar{\mathcal{L}} / \mathcal{X} - \left( \frac{d}{d\bar{l}} \bar{g}^{-2} \right) \frac{\delta \bar{l}}{\delta \mathcal{X}} \mathcal{X} \\
\bar{l} &= \frac{1}{8} \log \left( (\mathcal{X} / \mu^4)^2 \right)
\end{aligned} \tag{55}$$

$$\begin{aligned}
\frac{d}{d\bar{l}} \bar{g}^{-2} &= \frac{1}{8\pi^2} b(\bar{\kappa}) \\
b(\bar{\kappa}) &= -(\bar{g}\bar{\kappa})^{-1} \beta(\bar{g}) = -(b_0 + b_1 \bar{\kappa}^1 + \dots)
\end{aligned}$$

We substitute the derivatives from eq. 53

$$\frac{\delta \mathcal{X}}{\delta B_{\mu\nu}^r} = \frac{1}{2} B^{\mu\nu r} \tag{56}$$



which gives the derivatives of  $\mathcal{X}$ , using eq. 54

$$\begin{aligned}
\frac{\delta \mathcal{X}}{\delta (\partial_\varrho W_\sigma^s)} &= B^{\varrho \sigma s} ; \quad \frac{\delta \mathcal{X}}{\delta W_\sigma^s} = \frac{1}{2} B^{\mu\nu r} \frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} \\
\frac{\delta \mathcal{X}}{\delta W_\sigma^s} &= -f_{str} W_\varrho^t B^{\varrho \sigma r} \\
&= \frac{1}{2} B^{\mu\nu r} f_{srt} \left( \delta_\nu^\sigma W_\mu^t - \delta_\mu^\sigma W_\nu^t \right) \quad (\uparrow) \\
&= \frac{1}{2} f_{srt} \left( W_\mu^t B^{\mu \sigma r} - W_\nu^t B^{\sigma \nu r} \right) \quad (\uparrow) \\
\hline
\frac{\delta B_{\mu\nu}^r}{\delta (\partial_\varrho W_\sigma^s)} &= \delta^{rs} \left( \delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma \right) \\
\frac{\delta B_{\mu\nu}^r}{\delta W_\sigma^s} &= f_{srt} \left( \delta_\nu^\sigma W_\mu^t - \delta_\mu^\sigma W_\nu^t \right)
\end{aligned} \tag{57}$$

### 3 I - Insertion : Energy momentum tensor density as a conserved generalized Nöther current , restricted to $\overline{\mathcal{L}}$ in the absence of matter fields, i.e. neglecting $\mathcal{L}_{\{q\}}$

The Lagrangean density pertaining to quark-antiquark flavors  $\mathcal{L}_{\{q\}}$  is defined in eqs. 32 , 36 , the one for pure gauge fields  $\overline{\mathcal{L}}$ , induced by the trace anomaly , in eq. 50 .

We digress here, to derive the energy momentum density tensor  $\vartheta_\nu^\mu(x)$  as a local functional of the Lagrangean density  $\overline{\mathcal{L}}$  and the gauge field variables  $\partial_\varrho W_\sigma^s \longleftrightarrow B_{\varrho\sigma}^s$ ,  $W_\sigma^s$ , as they appear e.g. in eq. 57 in variational techniques going back to Emmy Nöther [5-1918] .

First we consider general variations of the base variables, understood to depend continuously on a family parameter, denoted  $f$ . This relates to the equations of motion

$$\begin{aligned}
W_\sigma^s &= W_\sigma^s(f; x) ; \quad \delta W_\sigma^s(x) = \partial_f W_\sigma^s(f; x) |_{f=0} \rightarrow \\
\delta \partial_\varrho W_\sigma^s(x) &= \partial_\varrho \delta W_\sigma^s(x)
\end{aligned} \tag{58}$$

and determine the conditions for the f-dependent action integral over a general 4-dimensional volume V

$$S(V, f) = \int_V d^4x \overline{\mathcal{L}}(W_\sigma^s(f, x); \partial_\varrho W_\sigma^s(f; x)) \tag{59}$$

to acquire an extremal value for the particular member of the family of base fields corresponding to  $f = 0$ .

This gives rise to the condition

$$\begin{aligned}
\delta S &= \partial_f S(V, f) \big|_{f=0} = 0 \rightarrow \\
\delta S &= \\
&= \int_V d^4 x \left( \delta W_\sigma^s \bar{\mathcal{L}}, W_\sigma^s \big|_{f=0} + \delta \partial_\varrho W_\sigma^s \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s \big|_{f=0} \right) \\
&= 0
\end{aligned}$$


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$$\begin{aligned}
\bar{\mathcal{L}}, W_\sigma^s \big|_{f=0} &\rightarrow \bar{\mathcal{L}}, W_\sigma^s = \frac{\partial \bar{\mathcal{L}}}{\partial W_\sigma^s(f=0; x)} \\
\bar{\mathcal{L}}, \partial_\varrho W_\sigma^s \big|_{f=0} &\rightarrow \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s = \frac{\partial \bar{\mathcal{L}}}{\partial (\partial_\varrho W_\sigma^s)(f=0; x)}
\end{aligned} \tag{60}$$

For simplicity of notation we neglect the symbol  $\big|_{f=0}$  in the quantities  $\bar{\mathcal{L}}, W_\sigma^s, \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s$  in the following as indicated in eq. 60.  $\delta S$  (eq. 60) becomes using eq. 58

$$\begin{aligned}
\delta S &= \int_V d^4 x \left( \delta W_\sigma^s \bar{\mathcal{L}}, W_\sigma^s + (\partial_\varrho \delta W_\sigma^s) \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s \right) \rightarrow \\
&(\partial_\varrho \delta W_\sigma^s) \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s = \\
&= \partial_\varrho (\delta W_\sigma^s \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s) - \delta W_\sigma^s \partial_\varrho \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s
\end{aligned} \tag{61}$$

Substituting the second relation in eq. 61 we obtain

$$\delta S = \int_V d^4 x \left[ \delta W_\sigma^s (\bar{\mathcal{L}}, W_\sigma^s - \partial_\varrho \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s) + \right] \tag{62}$$

The variational setting is incomplete without conditions on the boundary of the volume  $V$  – boundary conditions – which arise from the integration of the divergence in the last term in eq. 62

$$\begin{aligned}
&\int_V d^4 x \partial_\varrho (\delta W_\sigma^s \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s) = \\
&= \int_{\partial V} d^3 \sigma_{\varrho, \partial V} (\delta W_\sigma^s \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s) \rightarrow \\
&\rightarrow \text{boundary conditions : } \delta W_\sigma^s(x) = 0 \text{ for } x \in \partial V
\end{aligned} \tag{63}$$

With the boundary conditions as defined in eq. 63 satisfied, the extremum condition for the action integral (for all volumes  $V$ ) and *otherwise* arbitrary variations  $\delta W_\sigma^s(x)$  are equivalent to the local Euler-Lagrange equations of motion (eq. 62)

$$(\partial_\varrho \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s - \bar{\mathcal{L}}, W_\sigma^s)(x) = 0 \tag{64}$$

Eqs. 58 - 62 together with the boundary conditions in eq. 63 generate the Euler-Lagrange equations ( eq. 64 ) .

Together this sets the stage for deriving associating with every additional symmetry of the Lagrangean density a conserved Nöther 'current' , but with general spin, depending on the symmetry involved.

We apply the variations as generally given in eq. 58 to the special case relating rigid space-time translations to the canonical energy momentum tensor

$$\begin{aligned}
 x^\nu \rightarrow f a^\nu &\longleftrightarrow \begin{aligned} \delta_a W_\sigma^s &= a^\nu \partial_\nu W_\sigma^s \\ \delta_a \partial_\rho W_\sigma^s &= a^\nu \partial_\nu \partial_\rho W_\sigma^s \end{aligned} \\
 \hline
 \delta_a \bar{\mathcal{L}} &= a^\nu \left( \begin{aligned} &(\partial_\nu W_\sigma^s) \bar{\mathcal{L}},_{W_\sigma^s} + \\ &+ (\partial_\rho \partial_\nu W_\sigma^s) \bar{\mathcal{L}},_{\partial_\rho W_\sigma^s} \end{aligned} \right) = \\
 &= a^\nu \partial_\nu \bar{\mathcal{L}}
 \end{aligned} \tag{65}$$

The underlying translation symmetry reveals itself subtracting the last term in the expression for  $\delta_a \bar{\mathcal{L}}$  in eq. 65 from the second to yield

$$\begin{aligned}
 a^\nu \left( \begin{aligned} &\partial_\rho [ (\partial_\nu W_\sigma^s) \bar{\mathcal{L}},_{\partial_\rho W_\sigma^s} - \delta_\nu^\rho \bar{\mathcal{L}} ] - \\ &- (\partial_\nu W_\sigma^s) \mathcal{E}^{\sigma s} \end{aligned} \right) = 0 \\
 \mathcal{E}^{\sigma s}(x) &= (\partial_\rho \bar{\mathcal{L}},_{\partial_\rho W_\sigma^s} - \bar{\mathcal{L}},_{W_\sigma^s})(x) \rightarrow 0 \\
 &\text{Euler-Lagrange} \\
 &\text{equations}
 \end{aligned} \tag{66}$$

From eq. 66 we read off the canonical energy momentum density tensor in mixed components

$$\begin{aligned}
 T_\nu^\rho &= (\partial_\nu W_\sigma^s) \bar{\mathcal{L}},_{\partial_\rho W_\sigma^s} - \delta_\nu^\rho \bar{\mathcal{L}} ; \partial_\rho T_\nu^\rho = 0 \\
 T_{\nu\rho} &= \eta_{\rho\tau} T_\nu^\tau ; T_{\nu\rho} - T_{\rho\nu} \neq 0
 \end{aligned} \tag{67}$$

$T_{\nu\rho}$  is neither symmetric nor gauge invariant.

Nevertheless a symmetric and gauge invariant energy momentum density tensor can always be achieved, by also considering variations of  $\bar{\mathcal{L}}$  under Lorentz transformations.

The consistency of the gravitational coupling of  $\sqrt{g} \bar{\mathcal{L}}(g_{\mu\nu}; B_{\sigma\tau}^s)$ , with  $g = -\text{Det } g_{\mu\nu}$ , allows to simplify the explicit symmetrization procedure, due to Belinfante [6-1940], in the limit  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$ , i.e. in uncurved space time, after considering a variation of the metric to first order.

Here I follow my derivation in ref. [3-1981], denoting the symmetric energy

momentum tensor  $\vartheta_{\mu\nu}$ .

$$\begin{aligned}
2 \delta (\sqrt{g} \bar{\mathcal{L}}) &= \sqrt{g} \vartheta_{\mu\nu} (\delta g^{\mu\nu}) \big|_{g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}} \\
\delta (\sqrt{g} \bar{\mathcal{L}}) &= (\delta \sqrt{g}) \bar{\mathcal{L}} + \sqrt{g} \delta \bar{\mathcal{L}} \\
\delta : g^{\mu\nu} &\rightarrow g^{\mu\nu} + \delta g^{\mu\nu} ; [g_{\mu\nu} : \text{base metric}] \\
g^{\mu\nu} &= (g^{-1})_{\mu\nu} \\
\delta \sqrt{g} &= -\frac{1}{2} \sqrt{g} g_{\mu\nu} (\delta g^{\mu\nu})
\end{aligned} \tag{68}$$

In eq. 68  $\bar{\mathcal{L}}$  is, for general metric, a scalar quantity and  $\vartheta_{\mu\nu}$  a symmetric tensor.

The variation of  $\bar{\mathcal{L}}$  becomes using eq. 55

$$\begin{aligned}
\delta \bar{\mathcal{L}} &= \bar{\mathcal{L}}_{,X} \delta X ; X = \frac{1}{4} B_{\mu\sigma}^s B_{\nu\tau} g^{\mu\nu} g^{\sigma\tau} \\
\bar{\mathcal{L}}_{,X} &= \frac{\delta \bar{\mathcal{L}}}{\delta \mathcal{X}} = \bar{\mathcal{L}} / \mathcal{X} - \left( \frac{d}{d\bar{l}} \bar{g}^{-2} \right) \frac{\delta \bar{l}}{\delta \mathcal{X}} \mathcal{X} \\
\bar{l} &= \frac{1}{8} \log \left( (\mathcal{X} / \mu^4)^2 \right)
\end{aligned} \tag{69}$$

Proceeding step by step we first substitute

$$\frac{\delta \bar{l}}{\delta \mathcal{X}} \mathcal{X} = \frac{1}{4} ; \bar{\mathcal{L}} / \mathcal{X} = -(\bar{g}^{-2} - J) \tag{70}$$

in the second relation in eq. 69

$$\delta \bar{\mathcal{L}} = \left[ -(\bar{g}^{-2} - J) - \frac{1}{4} \left( \frac{d}{d\bar{l}} \bar{g}^{-2} \right) \right] \delta \mathcal{X} \tag{71}$$

Next we recall eqs. 50 - 51 and 55

$$\begin{aligned}
\frac{d}{d\bar{l}} \bar{g}^{-2} &= 2 (-\beta(\bar{g}) / \bar{g}^3) = \frac{1}{8\pi^2} b_0 B(\bar{\kappa}) \\
\bar{\kappa} &= \frac{\bar{g}^2}{16\pi^2} ; \bar{\alpha}_s = 4\pi \bar{\kappa}
\end{aligned} \tag{72}$$

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$$\begin{aligned}
B(\bar{\kappa}) &= b(\bar{\kappa}) / b_0 = [-\beta(\bar{g}) / \bar{g}] (b_0 \bar{\kappa})^{-1} \\
b_0 &= 11 - \frac{2}{3} N_f \\
B(\bar{\kappa}) &= B_0 + B_1 \bar{\kappa} + \dots \\
B_0 &= 1, B_n = b_n / b_0 ; n = 0, 1, 2, \dots
\end{aligned}$$

The differential equation in eq. 72 is to be solved for suitable initial conditions as if  $\bar{l}$ ,  $\bar{\kappa}$  were c-numbers and the functional dependence as indicated

in eq. 73 below evaluated with the operator valued (local field valued) substitution, defined in eq. 55

$$\bar{\kappa} = \bar{\kappa}(l) = F(l) ; l \longrightarrow \frac{1}{8} \log \left( (\mathcal{X} / \mu^4)^2 \right) \quad (73)$$

In the substitution, defined in eqs. 55 and 73, local products of local operators are involved, which again requires a regularisation in the sense of normal ordering of canonical variables .

We insert the relation in eq. 72 in eq. 71 and obtain

$$\delta \bar{\mathcal{L}} = \left[ - (\bar{g}^{-2} - J) - \frac{1}{32\pi^2} b_0 B(\bar{\kappa}) \right] \delta \mathcal{X} \quad (74)$$

Next we evaluate  $\delta \mathcal{X}$  using the first relation in eq. 55

$$\begin{aligned} \delta \bar{\mathcal{L}} &= \bar{\mathcal{L}},_X \delta X ; X = \frac{1}{4} B_{\mu\sigma}^s B_{\nu\tau} g^{\mu\nu} g^{\sigma\tau} \rightarrow \\ \delta X &= \frac{1}{2} B_{\mu\sigma}^s g^{\sigma\tau} B_{\nu\tau}^s \Big|_{g^{\alpha\beta} \rightarrow \eta^{\alpha\beta}} \delta g^{\mu\nu} \\ &= -\frac{1}{2} (B_{\mu\sigma}^s B_{\nu}^{\sigma s}) \delta g^{\mu\nu} \end{aligned} \quad (75)$$

The - sign in the last expression in eq. 75 together with a transposition of the tensor indices in  $B_{\nu}^{\sigma s}$  is chosen to offset the two - signs in the expression inside  $[\cdot]$  brackets in eq. 74 as well as to comply with the analogous expressions in QED [7-1976] .

Inserting the expressions in eq. 75 in eq. 74 the final form of  $\delta \bar{\mathcal{L}}$  becomes

$$\delta \bar{\mathcal{L}} = \frac{1}{2} \left[ (\bar{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\bar{\kappa}) \right] (B_{\mu\sigma}^s B_{\nu}^{\sigma s}) \delta g^{\mu\nu} \quad (76)$$

Finally we complete the expression for the symmetric energy momentum tensor  $\vartheta_{\mu\nu}$  in eq. 68 , which is conserved in the limit of uncurved space time

$$\begin{aligned} 2 \delta (\sqrt{g} \bar{\mathcal{L}}) &= \\ &= \sqrt{g} \vartheta_{\mu\nu} (\delta g^{\mu\nu}) \Big|_{g^{\alpha\beta} \rightarrow \eta^{\alpha\beta}} \rightarrow \\ \hline 2 \delta (\sqrt{g} \bar{\mathcal{L}}) &= \\ &= \sqrt{g} \left[ \left[ (\bar{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\bar{\kappa}) \right] B_{\mu\sigma}^s B_{\nu}^{\sigma s} \right. \\ &\quad \left. - \frac{1}{4} g_{\mu\nu} (\bar{g}^{-2} - J) B_{\varrho\sigma}^s B^{\sigma\varrho s} \right] \delta g^{\mu\nu} \end{aligned} \quad (77)$$

Combining the two terms proportional to  $\bar{g}^{-2} - J$  in eq. 77 we obtain two characteristic contributions to  $\sqrt{g} \vartheta_{\mu\nu} (\delta g^{\mu\nu})$  , even before the limit  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$  is taken .

It follows

$$\vartheta_{\mu\nu} = \left[ \begin{aligned} &(\bar{g}^{-2} - J) [B_{\mu\sigma}^s B_{\nu}^{\sigma s} - \frac{1}{4} g_{\mu\nu} B_{\varrho\sigma}^s B^{\sigma\varrho s}] \\ &+ \frac{1}{8\pi^2} b_0 B(\bar{\kappa}) [\frac{1}{4} B_{\mu\sigma}^s B_{\nu}^{\sigma s}] \end{aligned} \right] \quad (78)$$

The upper term in the outer  $[\cdot]$  brackets in eq. 78 , proportional to  $\bar{g}^{-2} - J$  , is traceless , whereas the anomalous trace is contained in the lower term , proportional to  $\frac{1}{8\pi^2} b_0 B(\bar{\kappa})$  .

In the limit of vanishing gravitational interactions , i.e.  $g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$  , the energy momentum tensor  $\vartheta_{\mu\nu}$  , defined in eq. 78 , is not only symmetric but conserved , provided the *modified* equations of motion can be enforced, as well as gauge invariant with respect to the local gauge group of QCD  $SU3_c$

$$\partial^\mu \vartheta_{\mu\nu}(x) = 0 ; \quad \partial^\mu = \eta^{\mu\beta} \partial_\beta = \eta^{\mu\beta} \partial / \partial x_\beta \quad (79)$$

The trace anomaly allows precisely the general form of the modified Lagrangean  $\bar{\mathcal{L}}$  , as introduced in eq. 50 including the a priori free constant  $J$  . It takes the form using eq. 78 , independent of  $J$

$$\begin{aligned} \bar{\mathcal{L}} &= - \left[ \frac{1}{\bar{g}^2(\mathcal{X})} - J \right] \mathcal{X} \rightarrow \\ \vartheta^\mu_\mu &= \frac{1}{8\pi^2} b_0 B(\bar{\kappa}) \left[ \frac{1}{4} B^s_{\mu\sigma} B^{\sigma\mu s} \right] \\ &= - \frac{1}{8\pi^2} b_0 : B(\bar{\kappa}(\mathcal{X})) \mathcal{X} : \end{aligned} \quad (80)$$

In the last expression in eq. 80 the need of ( normal ) ordering of local products of quantized fields is indicated by the  $:$  symbols , omitted before and also hereafter whenever not explicitly necessary .

The negative sign in the same expression is significant: the factor  $B(\bar{\kappa})(\mathcal{X})$  is a positive operator at least for  $\bar{\kappa}$  in the perturbative regime, while  $\mathcal{X}$  becomes a positive operator in the Euclidean region

$$\begin{aligned} \mathcal{X} &= \frac{1}{2} \left[ \vec{B}^s \vec{B}^s - \vec{E}^s \vec{E}^s \right] \rightarrow \\ &\rightarrow \mathcal{X}_{Euc} = \frac{1}{2} \left[ \vec{B}^s \vec{B}^s + \vec{E}^s \vec{E}^s \right]_{Euc} \end{aligned} \quad (81)$$

This completes the construction of the trace anomaly induced canonical variables and modified Lagrangean  $\bar{\mathcal{L}}$  . Consequences are discussed in the next subsection.

### 3 C - Remarks and consequences arising from derivations in the last subsection

$$\left( \begin{array}{l} \text{3 I - Insertion : Energy momentum tensor density as a} \\ \text{conserved generalized Nöther current} \\ \text{restricted to } \bar{\mathcal{L}} \text{ in the absence} \\ \text{of matter fields} \\ \text{i.e. neglecting } \mathcal{L}_{\{q\}} \end{array} \right)$$

- 1) Remark relative to the symmetric energy momentum tensor compatible with the trace anomaly

while the energy momentum tensor as defined in eq. 78 is symmetric by the consistency of the embedding into gravitational interactions

leading to the variational form given in eq. 68 , it is only conserved if the equations of motions or Euler-Lagrange equations relative to a specific Lagrangean density –  $\mathcal{L}_{general}$  – are satisfied, i.e. if the relations , restricted to gauge fields in eq. 68

$$\begin{aligned} \mathcal{E}^{\sigma s}(x) &= (\partial_\varrho \bar{\mathcal{L}}, \partial_\varrho W_\sigma^s - \bar{\mathcal{L}}, W_\sigma^s)(x) \rightarrow 0 \\ &\text{Euler-Lagrange} \\ &\text{equations} \\ \bar{\mathcal{L}} &\rightarrow \mathcal{L}_{general} ; \text{ e.g. } \rightarrow \mathcal{L}_{gauge} = -\frac{1}{g^2} \mathcal{X} \end{aligned} \quad (82)$$

$$\mathcal{X} = \frac{1}{4} B_{\mu\nu}^r B^{\mu\nu r}$$

are satisfied , yet for any gauge invariant Lagrangean, depending only on gauge potentials and their first derivatives. In particular substituting ( back )  $\bar{\mathcal{L}} \rightarrow \mathcal{L}$  , defined in eq. 50 , as indicated in the last relation in eq. 82 .

We now distinguish both the canonical energy momentum ( density ) tensors ( eq. 67 ) and their symmetric equivalents according to the two choices for the gauge field Lagrangean ( density ) ( eq. 77 ) , abbreviated to  $\mathcal{L}_{gauge} = \mathcal{L}$  and  $\bar{\mathcal{L}}$  respectively

$$\begin{aligned} T_\nu^\varrho &= (\partial_\nu W_\sigma^s) \mathcal{L}_{,\partial_\varrho W_\sigma^s}^{general} - \delta_\nu^\varrho \mathcal{L}^{general} \\ 2\delta(\sqrt{g} \mathcal{L}^{general}) &= \sqrt{g} \vartheta_{\mu\nu}(\delta g^{\mu\nu}) \big|_{g_{\alpha\beta} \rightarrow \eta_{\alpha\beta}} \rightarrow \\ \hline T_\nu^\varrho &= T_\nu^\varrho(\cdot) ; \vartheta_{\mu\nu} = \vartheta_{\mu\nu}(\cdot) ; \text{ with } \cdot = \{ \bar{\mathcal{L}}, \mathcal{L} \} \end{aligned} \quad (83)$$

The Belinfante construction [6-1940] links uniquely  $T_\nu^\mu(\cdot) \leftrightarrow \vartheta_{\mu\nu}(\cdot)$  , separately for both values of  $(\cdot)$  . The respective canonical energy momentum tensors are neither symmetric nor gauge invariant also for both choices of  $(\cdot)$  , whereas both  $\vartheta_{\mu\nu}(\cdot)$  are gauge invariant.

Furthermore all energy momentum tensors are conserved , provided the equations of motion are enforced

$$\begin{aligned} \partial^\mu T_{\nu\mu}(\cdot) &= 0 ; T_{\nu\mu}(\cdot) = \eta_{\mu\varrho} T_\nu^\varrho(\cdot) \\ \partial^\mu \vartheta_{\nu\mu}(\cdot) &= 0 \end{aligned} \quad (84)$$

Notwithstanding the two cases it is  $\bar{\mathcal{L}}$  which needs to be chosen. With respect to the trace anomaly ( eq. 80 ) and thereby with respect to dilatation transformations, for which the symmetric tensor(s)  $\vartheta_{\nu\mu}(\cdot)$  are the relevant ones, it follows

$$\begin{aligned} \vartheta_\mu^\mu(\bar{\mathcal{L}}) &= \frac{1}{8\pi^2} b_0 B(\bar{\kappa}) \left[ \frac{1}{4} B_{\mu\sigma}^s B^{\sigma\mu s} \right] \\ &= -\frac{1}{8\pi^2} b_0 : B(\bar{\kappa}(\mathcal{X})) \mathcal{X} : \neq 0 \\ \vartheta_\mu^\mu(\mathcal{L}) &= 0 \end{aligned} \quad (85)$$

From  $\vartheta_{\nu\mu}(\cdot)$  the canonical form the local dilatation current is constructed

$$\begin{aligned} d_{\mu}(\cdot)(x) &= (x - x_{(0)})^{\nu} \vartheta_{\nu\mu}(\cdot)(x) \\ \partial^{\mu} d_{\mu}(\cdot)(x) &= \vartheta^{\mu}_{\mu}(\cdot)(x) \\ \partial^{\mu} d_{\mu}(\overline{\mathcal{L}}) &\neq 0 ; \partial^{\mu} d_{\mu}(\mathcal{L}) = 0 \end{aligned} \quad (86)$$

In the case of  $\mathcal{L}$  the dilatation current is conserved and thus dilatation symmetry is enforced and as a direct consequence the entire group of conformal space time transformations becomes a symmetry group.

It is however  $\overline{\mathcal{L}}$  which has to be chosen as the only case compatible with the infrared regularity conditions inherent to maintaining local gauge transformations exact through the infrared unstable region , as discussed in ref. [1-2011] .

2) The new content of the last subsection dates from December 2011

The material presented here mainly represents a traceback of aspects of QCD , having been noted but not carried out in the past in any detail and characterized ( best ) as 'work in progress' .

This topic concerns the embedding of Hamiltonian quantum mechanics and canonically conjugate variables within local field theory into the predominantly perturbative treatments derived from asymptotic freedom in the ultraviolet of QCD .

Some bridges ahead of completion go back to my contributions to two events organized by Harald Fritzsch and the Nanyang Technological University, Singapore :

a) Conference in Honour of Murray Gell-Mann's 80th Birthday

'Quantum Mechanics, Elementary Particles, Quantum Cosmology and Complexity',  
24.-26. February 2010

b) International Conference on Flavor physics in the LHC era

8. - 12. November 2010

both held at the Nanyang Executive Centre in Singapore .

1

### 3 C 1 - Equations of motion and canonically conjugate variables pertaining to $\overline{\mathcal{L}} + \mathcal{L}_{\{q\}}$

The conflict between the canonical forms of obviously inequivalent Lagrangean densities in the gauge field sector

$$\overline{\mathcal{L}} \longleftrightarrow \mathcal{L} \big|_{gauge\,fields} \quad (87)$$

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<sup>1</sup> My gratitude goes to Professor Kokk Koo Phua and the organizing committees for making these events possible .



can be traced – in the perturbative sector – to maintaining Poincaré invariance and full gauge invariance , which necessitates the use of Fermi like gauges and thus a nontrivial coupling of ghost fields, which do contribute to the energy momentum tensor [8-1975] .

$$\mathcal{L} = \left[ \bar{q}_{\mathcal{A}' f}^{\dot{c}'} \left\{ \begin{array}{l} \frac{i}{2} \overleftrightarrow{\partial}_{\mu} \delta_{c'c} \\ - v_{\mu}^s \left( \frac{1}{2} \lambda^s \right)_{c'\dot{c}} \\ - m_f \bar{q}_{\dot{\mathcal{A}} f}^{\dot{c}} q_{\mathcal{A} f}^c \end{array} \right\} \gamma_{\mathcal{A}' \mathcal{A}}^{\mu} q_{\mathcal{A} f}^c \right] ; v_{\mu}^s = - W_{\mu}^s$$

quarks :  $c' , c = 1, 2, 3$  color ,  $f = 1, \dots, 6$  flavor  
 $\mathcal{A}', \mathcal{A} = 1, \dots, 4$  spin ,  $m_f$  mass

gauge bosons :

$$B_{\mu\nu}^r = \partial_{\mu} W_{\nu}^r - \partial_{\nu} W_{\mu}^r + f_{rst} W_{\mu}^s W_{\nu}^t$$

$r, s, t = 1, \dots, \dim (G = SU3_c) = 8$

Lie algebra labels,  $[\frac{1}{2} \lambda^r , \frac{1}{2} \lambda^s] = if_{rst} \frac{1}{2} \lambda^t$

perturbative rescaling :

$$W_{\mu}^r = g W_{\mu \text{ pert}}^r , B_{\mu\nu}^r = g B_{\mu\nu \text{ pert}}^r$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.  
Completing  $\Delta \mathcal{L}$  in Fermi gauges

$$\Delta \mathcal{L} = \left\{ \begin{array}{l} - \frac{1}{2 \eta g^2} ( \partial_{\mu} W^{\mu s} )^2 \\ + \partial^{\mu} \bar{c}^s ( D_{\mu} c )^s \end{array} \right\} ; \eta : \text{gauge parameter}$$

ghost fermion fields :  $c , \bar{c} ; ( D_{\mu} c )^r = \partial_{\mu} c^r + f_{rst} W_{\mu}^s c^t$   
gauge fixing constraint :  $C^r = \partial_{\mu} W^{\mu r}$

After these preliminary remarks we turn to the equations of motion for gauge fields as induced by the Lagrangean  $\bar{\mathcal{L}} + \mathcal{L}_{\{q\}}$  defined in eqs. 37 , 50 ( and 88 ) .

We merge eqs. 57 and 74

$$\frac{\delta \bar{\mathcal{L}}}{\delta \mathcal{X}} = - \left[ ( \bar{g}^{-2} - J ) + \frac{1}{32 \pi^2} b_0 B ( \bar{\kappa} ) \right] \frac{\delta \mathcal{X}}{\delta ( \partial_{\varrho} W_{\sigma}^s )} = B^{\varrho \sigma s} ; \frac{\delta \mathcal{X}}{\delta W_{\sigma}^s} = - f_{str} W_{\varrho}^t B^{\varrho \sigma r} \quad (91)$$

It follows for the partial derivatives of  $\overline{\mathcal{L}}$

$$\begin{aligned} \overline{\mathcal{L}}, \partial_\varrho W_\sigma^s = & + \left[ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}) \right] B^{\sigma\varrho s} \\ \overline{\mathcal{L}}, W_\sigma^s = & - \left[ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}) \right] f_{str} \times \\ & \times W_\varrho^t B^{\sigma\varrho r} \end{aligned} \quad (92)$$

Using eq. 48

$$(D_\varrho(ad))_{sr} \{B_{\sigma\tau}^r\} = \partial_\varrho \{B_{\sigma\tau}^s\} + f_{str} W_\varrho^t \{B_{\sigma\tau}^r\} \quad (93)$$

the Euler-Lagrange derivative of  $\overline{\mathcal{L}}$  on the left hand side of eq. 64 becomes

$$\begin{aligned} (\partial_\varrho \overline{\mathcal{L}}, \partial_\varrho W_\sigma^s - \overline{\mathcal{L}}, W_\sigma^s)(x) = \\ = D_\varrho(ad) \left\{ \left[ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}) \right] B^{\sigma\varrho} \right\}^s(x) \end{aligned} \quad (94)$$

The Euler-Lagrange equations including quark flavors according to  $\mathcal{L}_{\{q\}}$  (eqs. 35, 40, 88) take the form

$$\begin{aligned} D_\varrho(ad) \left\{ \left[ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}) \right] B^{\sigma\varrho} \right\}^s(x) = \\ = (j^{\sigma s})_{\{q\}}(x) \end{aligned} \quad (95)$$

---


$$(j^{\sigma s})_{\{q\}} = \sum_{q-fl} \overline{q}^{\dot{c}'} \left\{ \gamma^\sigma \left( \frac{1}{2} \lambda^s \right)_{\dot{c}'\dot{c}} \right\} q^c = (\mathcal{L}_{\{q\}})_{, W_\sigma^s}$$

The quantity on the gauge field side of the 'divergence' on the right hand side of eq. 94 and on the left hand side of the first relation in eq. 95 contrasts in an essential way with the bare Lagrangean expressions corresponding to  $\mathcal{L}_{gauge}$  (eqs. 40, 107) This is the main and new result of these last subsections.

To make this more transparent lets introduce the notation

$$\begin{aligned} \overline{\mathcal{L}} &\leftrightarrow \overline{G} && \leftrightarrow \mathcal{L} \leftrightarrow G \\ \overline{G} = \text{local, scalar, color neutral field} &\leftrightarrow G = && g^{-2} \\ &&& \text{const. c-number} \\ \overline{G} =: & \left[ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}) \right] : (x) \end{aligned} \quad (96)$$

The precise way,  $\overline{G}$  and associated quantities, in particular the energy momentum (density) tensor pertaining to gauge fields, defined in eq. 78, are determined as local field operatoras, shall be (re-) specified below

$$\begin{aligned} (a) : \overline{G} &= \overline{G}(l); l = \log(\overline{\mu}/\mu) \rightarrow \\ (b) : l &\rightarrow \frac{1}{8} \log \left( (\mathcal{X}(x)/\mu^4)^2 \right) \leftrightarrow e^{8l} = (\mathcal{X}(x)/\mu^4)^2 \end{aligned}$$

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$$\mathcal{X}(x) =: \frac{1}{4} B_{\mu\nu}^r B^{\mu\nu r} : (x) \quad (97)$$

The functional dependence  $\overline{G} = \overline{G}(l)$  in step (a) in eq. 97 can be determined in the perturbative regime, through the renormalization group equation(s), e.g. in the  $\overline{MS}$  renormalization scheme.

Through 4 loop order  $\overline{MS}$  is renormalization group invariant, through the substitutions  $\mu = \Lambda_{QCD}$  and e.g. through the moments of deep inelastic scattering amplitudes,  $\overline{\mu}^2 = Q^2$ , where  $Q^2$  is the (positive) momentum transfer square in the deep inelastic reaction studied [10-1988, 11-1997].

In any renormalization scheme, where at least in the perturbative regime full renormalization group invariance is verified, it follows that the substitution in the second step (b) in eq. 97

$$l \rightarrow \frac{1}{8} \log \left( (\mathcal{X}(x) / \mu^4)^2 \right) \leftrightarrow e^{8l} = (\mathcal{X}(x) / \mu^4)^2 \quad (98)$$

can equally be performed in a renormalization group invariant manner. This is tantamount to resolve all ambiguities in the definition of the composite local field  $\mathcal{X}(x)$  by a normalization in terms of renormalization group invariant, i.e. measurable quantities.

Within QCD sum rules introduced by Shifman, Vainshtain and Zakharov [12-1979], the vacuum expected value of the multiplicatively related field, denoted  $\alpha_s G^2$  has been intensively studied.

I cite here a recent paper and result(s) by Stephan Narison [13-2011] in particular with respect to the renormalization group invariant setting – in principle – of composite local field normalization

$$\begin{aligned} \alpha_s G^2 &= \pi^{-1} \mathcal{X} \\ \langle \Omega | \alpha_s G^2 | \Omega \rangle &= (7.0 \pm 1.3) 10^{-2} \text{ GeV}^4 \\ &= \pi^{-1} (0.22 \pm 0.04) \text{ GeV}^4 \end{aligned} \quad (99)$$

Having defined the local field structure of two inequivalent such fields  $\mathcal{X}(x)$  and *separately*  $\overline{G}(x)$  in eqs. 96 and 97, the symmetric, gauge invariant energy momentum tensor incompletely defined in eqs. 77 and 78 is represented as follows

$$\begin{aligned} \vartheta_{\mu\nu}(x) &= \left[ \begin{array}{c} : \overline{G} [ B_{\mu\sigma}^s B_{\nu}^{\sigma s} + g_{\mu\nu} \mathcal{X} ] : \\ - g_{\mu\nu} : \frac{1}{32\pi^2} b_0 B(\overline{\kappa}(\mathcal{X})) \mathcal{X} : \end{array} \right] (x) \\ \vartheta_{\mu}^{\mu}(x) &= - \frac{1}{8\pi^2} b_0 : B(\overline{\kappa}(\mathcal{X})) \mathcal{X} : (x) \\ \hline \overline{G}(x) &= : [ (\overline{g}^{-2} - J) + \frac{1}{32\pi^2} b_0 B(\overline{\kappa}(\mathcal{X})) ] : (x) \\ B(\overline{\kappa}(\mathcal{X}))(x) &\rightarrow \equiv \overline{B}(\mathcal{X})(x) \end{aligned} \quad (100)$$

In eq. 100  $B(\overline{\kappa}(\mathcal{X}))$  indicates that through the substitution in step (b) of  $\overline{G}$  in eq. 97, each of the additive terms of  $\overline{G}$  and thus also  $B(\overline{\kappa}) \rightarrow \overline{B}(\mathcal{X})$  becomes implicitly dependent on  $\mathcal{X}$ .

Next we complete the Euler-Lagrange equations of motion defined in eqs. 94 and 95 , which introduce a local, color neutral, hermitian scalar field , denoted  $\varphi ( x )$  below , depending implicitly on the gauge field strengths bilinear  $\mathcal{X} ( x )$  as specified in eqs. 96 - 98

$$\begin{aligned} & ( \partial_\varrho \overline{\mathcal{L}} , \partial_\varrho W_\sigma^s - \overline{\mathcal{L}} , W_\sigma^s ) ( x ) = \\ & = D_\varrho ( ad ) : \left\{ \left[ \left( \overline{g}^{-2} - J + \frac{1}{32\pi^2} b_0 \overline{B} \right) ( \mathcal{X} ) \right] B^{\sigma\varrho} \right\}^s : ( x ) \rightarrow \\ & \varphi ( x ) = : ( \overline{g}^{-2} - J + \frac{1}{32\pi^2} b_0 \overline{B} ) ( \mathcal{X} ) : ( x ) \equiv \overline{G} ( x ) \end{aligned} \quad (101)$$

The field  $\varphi$  introduced in eq. 101 is **dimensionless** , which in itself is a consequence of the violation of dilatation invariance, or the trace anomaly. The equations of motion for the gauge fields ( eq. 95 ) become , suppressing the ordering signs :

$$\begin{aligned} & D_\varrho ( ad ) \{ \varphi B^{\sigma\varrho} \}^s ( x ) = ( j^{\sigma s} )_{\{q\}} ( x ) \\ & \overline{( j^{\sigma s} )_{\{q\}}} = \sum_{q-fl} \overline{q}^{\dot{c}'} \{ \gamma^\sigma ( \frac{1}{2} \lambda^s )_{\dot{c}'\dot{c}} \} q^{\dot{c}} = ( \mathcal{L}_{\{q\}} ) , W_\sigma^s \\ & \begin{aligned} D_\varrho ( ad ) \\ \{ \varphi B^{\sigma\varrho} \}^s \\ ( x ) \end{aligned} &= \begin{bmatrix} \partial_{x\varrho} \{ \varphi ( x ) B^{\sigma\varrho s} ( x ) \} + \\ + f_{str} \{ W_\varrho^t ( x ) \varphi ( x ) B^{\sigma\varrho r} ( x ) \} \end{bmatrix} \\ & &= \begin{bmatrix} \{ \partial_{x\varrho} \varphi ( x ) \} \{ B^{\sigma\varrho s} ( x ) \} + \\ + \varphi ( x ) \{ D_\varrho ( ad ) B^{\sigma\varrho} \}^s ( x ) \end{bmatrix} \\ & \overline{\varphi ( x )} = : ( \overline{g}^{-2} - J + \frac{1}{32\pi^2} b_0 \overline{B} ) ( \mathcal{X} ) : ( x ) \end{aligned} \quad (102)$$

In deriving eq. 102 we assumed that chain rules for normal partial derivatives and covariant derivatives are maintained through the ordering processes , suppressed for simplicity of notation .

### 3 CD - Equations of motion modify the canonically conjugate variables pertaining to $\overline{\mathcal{L}}$ , beyond the reduction to consider exclusively the composite local field $\mathcal{X} ( x )$

Finally we turn to the consequences elaborated in the previous subsection , as they arise for the structure of canonically conjugate variables , in an axial gauge  $W_0^s ( x ) = 0$  , in *essential* contrast to the structure pertaining to the bare Lagrangian  $\mathcal{L}$  , as discussed in the section **3 a - Bare Lagrangian density and equations of motion in unconstrained gauges** .

Thus we consider the canonically conjugate variables pertaining to  $\overline{\mathcal{L}}$

$$W_\sigma^s \leftrightarrow \overline{\mathcal{L}} , \partial_0 W_\sigma^s \rightarrow \sigma = m = 1, 2, 3 \text{ for } W_0^s = 0 \quad (103)$$

The derivatives of  $\overline{\mathcal{L}}$  are given in eq. 92 . Using the scalar field  $\varphi$  , defined in eq. 101 , eq. 92 takes the form

$$\begin{aligned}\overline{\mathcal{L}}_{, \partial_\varrho W_\sigma^s} &= \varphi B^{\sigma \varrho s} \\ \overline{\mathcal{L}}_{, W_\sigma^s} &= -f_{str} W_\varrho^t \varphi B^{\sigma \varrho r}\end{aligned}\tag{104}$$


---


$$\varphi(x) = : \left( \overline{g}^{-2} - J + \frac{1}{32\pi^2} b_0 \overline{B} \right) (\mathcal{X}) : (x)$$

From eq. 104 we obtain the canonically conjugate momentum fields relative to  $W_{m=\sigma}^s$

$$\mathcal{A} : \overline{\Pi}^{ms} = \varphi B^{m0s} = -\varphi \left( \vec{E}^s \right)^m ; m = 1, 2, 3 \tag{105}$$

to be compared with eq. 44 relative to  $\mathcal{L}$  .

Using the substitution  $\overline{\Pi} \leftrightarrow \overline{\mathcal{L}}$  , the Ansatz for equal time commutation relations in eq. 45 remains the same

$$\begin{aligned}\mathcal{A} : \left[ W_m^r(t, \vec{x}), \overline{\Pi}^{sn}(t, \vec{y}) \right] &= i \delta^{rs} \delta_m^n \delta^{(3)}(\vec{x} - \vec{y}) \P \\ \left[ W_m^r(t, \vec{x}), W_n^s(t, \vec{y}) \right] &= 0 \\ \left[ \overline{\Pi}^{rm}(t, \vec{x}), \overline{\Pi}^{sn}(t, \vec{y}) \right] &= 0\end{aligned}\tag{106}$$

yet the intervention of the nontrivial scalar field  $\varphi$  – very inequivalent from the bare Lagrangean counterpart  $g^{-2}$  – shows that consistency of the Euler-Lagrange equations takes its toll, in an interesting way. This result is the key advance I can report here and now .

In this and the next subsection it remains to draw some conclusions and sketch eventual future advances .

### 3 D - Remarks and consequences arising from derivations in the last subsections : 3I , 3C , 3C1 , 3CD

- 1) The original derivation of the trace anomaly was done in the perturbative region in the ultraviolet for QCD and in the infrared for QED . This was sufficient for QCD , to establish through dominance of the leading contribution  $2\beta(g)/g \left( \frac{1}{4} B_{\mu\nu}^{s\,pert.} B^{\mu\nu\,s\,pert.} \right)$  over quark mass terms , the breaking of dilatation invariance for vanishing quark masses.

Nevertheless – as far as the overall dominance is concerned – for *all* scales , extending to the nonperturbative and unstable regions , I have not checked before , whether the unique renormalization and normalization of the field strength bilinear  $\mathcal{X} = \frac{1}{4} B_{\mu\nu}^s B^{\mu\nu\,s}$  was sufficient to yield the ultraviolet limiting form of the trace of the symmetric energy momentum tensor  $\vartheta^\mu_\mu$  modulo a numerical constant related

to the rescaling function  $\beta(g)$  and vanishing only for identically vanishing  $\beta$ .

However there persisted an inconsistency between singling out the composite local field  $\mathcal{X}$  as the only contribution to the trace anomaly, since the so restricted Euler-Lagrange equations of motion did not satisfy the anomalous trace Ward identities, rather yield a traceless energy momentum tensor.

This leads as shown in the subsections 3I, 3C, 3C1, 3CD to a very different structure of the full trace anomaly, whereby the above inconsistency is resolved.

In the language of e.g. the QCD sum rules, in which the perturbative ordering of composite local operators according to their mass ( $M^n$ ) dimension –  $n$  – is essential in the perturbative regime, this ordering is upset in the full trace anomaly, whence extended to all physical scales, and mixing of essentially all  $n$  dimensional operators, remaining local does occur. The high  $n$  fields do not enter with arbitrarily large numbers of derivatives, endangering locality, but in a canonical Hamiltonian sense through high powers of only zero'th and first (covariant) derivatives of the base fields of QCD.

This is, a new result, borne out in the appearance in the equations of motion of the composite scalar and gauge invariant local field  $\varphi(x)$  as defined in eqs. 92 - 104, in the subsections 3C1, 3CD.  $\varphi$  is dependent on the basic fields of QCD and carries mass dimension 0.

- 2) Generating consistent second order (Hamiltonian) equations through a variational principle

In a way the use of generalized Nöther currents [5-1918] modulo Euler-Lagrange equations is here reversed. The relations derived are consistency equations between the trace of the energy momentum density tensor and the local field operators equal to it. They derive from a Hamiltonian system and its Euler-Lagrange equations only, in contrast to quark current algebra relations for additional flavor symmetries.

- 3) In the construction of the full symmetric and gauge invariant energy momentum density tensor components, also gravitational interactions enter, albeit only in the uncurved space limit  $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$  as shown in eq. 68. The complications in enforcing strict local gauge invariance – with respect to  $SU3_c$  here – have a clear basis in the Becchi, Rouet, Stora and Tyutin construction of BRST transformation rules for ghost fields [14-1975, 15-1975], necessarily present in Lorentz covariant gauges.

The nonperturbative completion of the full field theories in question is a separate far goal, beyond the scope of the present discussion .

— Thank you —

Acknowledgement : I would like to thank Raymond Stora , Rod Crewther and Andrei Kataev for discussions , even if in the late stages of this work they were only 'preliminary' as far as the topics presented here are concerned .

### 3 b - Dimension $[ M^4 ]$ equations for field strengths in unconstrained gauges

The following material was derived before the main discussion of the insertions : sections 3I to 3D .

The equations of motion ( eq. 40 ) as well as the Bianchi identities to which we turn below are of engineering dimension  $[ M^3 ]$  . In order to derive the dimension  $[ M^4 ]$  equations we introduce the matrix notations as adapted to the adjoint representation and convert Lorentz indices freely between covariant and contravariant ones

$$\begin{aligned} D_\varrho &= \partial_\varrho + \mathcal{W}_\varrho ; \mathcal{W}_\varrho = W_\varrho^r ad_r \\ D^\varrho &\left\{ \frac{1}{g^2} B_{\sigma\varrho} \right\}^s = j_\sigma^s \end{aligned} \quad (107)$$

Then we form the antisymmetric covariant ( electric- and magnetic- ) dipole current densities and suppress the adjoint component  $^s$  and abbreviate the inverse square bare coupling constant by  $G = g^{-2}$

$$\begin{aligned} G &= g^{-2} \\ D_\tau D_\varrho \{ G B_\sigma^\varrho \} - D_\sigma D_\varrho \{ G B_\tau^\varrho \} &= D_\tau j_\sigma - D_\sigma j_\tau \end{aligned} \quad (108)$$

Next we move the covariant derivatives  $D_\tau$  ,  $D_\sigma$  to the right using the identity

$$D_\tau D_\varrho = [ D_\tau , D_\varrho ] + D_\varrho D_\tau \quad (109)$$

Eq. 108 thus takes the form

$$\begin{aligned} \{ [ D_\tau , D_\varrho ] + D_\varrho D_\tau \} \{ G B_\sigma^\varrho \} - ( \tau \leftrightarrow \sigma ) &= \\ = D_\tau j_\sigma - D_\sigma j_\tau \end{aligned} \quad (110)$$

The commutators of covariant derivatives reduce to the adjoint field strength matrix valued form

$$[ D_\tau , D_\varrho ] = \mathcal{B}_{\tau\varrho} = B_{\tau\varrho}^t ad_t ; \tau\varrho \rightarrow \sigma\varrho \quad (111)$$

which yields , substituted into eq. 110

$$\left[ \begin{aligned} & D^{\varrho} ( D_{\tau} B_{\sigma\varrho} - D_{\sigma} B_{\tau\varrho} ) + \\ & + \mathcal{B}_{\tau\varrho} B_{\sigma}^{\varrho} - \mathcal{B}_{\sigma\varrho} B_{\tau}^{\varrho} \end{aligned} \right] = g^2 ( D_{\tau} j_{\sigma} - D_{\sigma} j_{\tau} ) \quad (112)$$

At this stage whence factoring out  $G$  beyond partial derivatives in eq. 112 we assume that  $G$  is not space-time dependent, and come back at a later stage to consider an arbitrary space-time dependent extension

$$\partial_{\mu} G = 0 \text{ with the extension } G \rightarrow \tilde{G}(x) \text{ with } \lim_{x \rightarrow \infty} \tilde{G}(x) = G \quad (113)$$

as an external source with appropriate boundary conditions for  $x \rightarrow \infty$ . The quantity in brackets in the first line of the left hand side of eq. 112 can be transformed , using the Bianchi identity [1-2011] for the covariant derivatives of field strengths

$$\begin{aligned} D_{\tau} B_{\sigma\varrho} - D_{\sigma} B_{\tau\varrho} &= D_{\tau} B_{\sigma\varrho} + D_{\sigma} B_{\varrho\tau} = -D_{\varrho} B_{\tau\sigma} \longrightarrow \\ \text{Bianchi identity : } D_{\tau} B_{\sigma\varrho} + D_{\sigma} B_{\varrho\tau} + D_{\varrho} B_{\tau\sigma} &= 0 \end{aligned} \quad (114)$$

The dipole density second order differential equation thus can be brought to the form

$$\left[ \begin{aligned} & D^{\varrho} D_{\varrho} B_{\sigma\tau} + \\ & + \mathcal{B}_{\tau\varrho} B_{\sigma}^{\varrho} - \mathcal{B}_{\sigma\varrho} B_{\tau}^{\varrho} \end{aligned} \right] = g^2 ( D_{\tau} j_{\sigma} - D_{\sigma} j_{\tau} ) \quad (115)$$

We summarize the equations of motion – eqs. 39 , 107 – and the chromo-dipole-density  $\dim [M^4]$  derived equations – eq. 115 – below

$\partial_{\varrho} \left\{ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta (\partial_{\varrho} W_{\sigma}^s)} \right\} - \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^r} \frac{\delta B_{\mu\nu}^r}{\delta W_{\sigma}^s} = \frac{\delta \mathcal{L}_{\{q\}}}{\delta W_{\sigma}^s}$ $\frac{\delta \mathcal{L}_{\{q\}}}{\delta W_{\sigma}^s} = \sum_{q-fl} \bar{q}^{\dot{c}'} \left\{ \gamma^{\sigma} \left( \frac{1}{2} \lambda^s \right)_{\dot{c}'\dot{c}} \right\} q^{\dot{c}} = (j^{\sigma s})_{\{q\}}$
$D_{\varrho} = \partial_{\varrho} + \mathcal{W}_{\varrho} ; \mathcal{W}_{\varrho} = W_{\varrho}^r ad_r$ $D^{\varrho} \{ B_{\sigma\varrho} \}^s = g^2 j_{\sigma}^s$
$\left[ \begin{aligned} & D^{\varrho} D_{\varrho} B_{\sigma\tau} + \\ & + \mathcal{B}_{\tau\varrho} B_{\sigma}^{\varrho} - \mathcal{B}_{\sigma\varrho} B_{\tau}^{\varrho} \end{aligned} \right] = g^2 ( D_{\tau} j_{\sigma} - D_{\sigma} j_{\tau} )$

(116)



### 3 c - QED : bare Lagrangean density and equations of motion in unconstrained – abelian – gauges

I consider it worth the effort to align all conventions to the (an) extension of QCD to the only other unbroken charge like gauge field theory : QED . To this end we extend the equations of motion accordingly starting with connections in eq. 1 in section 2 and eq. 32 in subsection 3a , as follows

$$\begin{aligned} \mathcal{G} &\rightarrow SU3_c \times U1_{em} ; q \rightarrow f = \begin{pmatrix} \updownarrow q_c \\ \updownarrow \ell \end{pmatrix} \\ \mathcal{W}_\mu &\rightarrow W_\mu^r d_r(\mathcal{D}) + W_\mu^{em} d_{em} \\ \mathcal{D} &: \begin{bmatrix} \text{irreducible} \\ \text{representation} \\ \text{of } SU3_c \\ \text{acting on q flavors only} \end{bmatrix} ; d_{em} : [\text{diagonal matrix}] \end{aligned} \quad (117)$$

The  $\updownarrow$  arrows in the column vector  $f_\bullet$  in eq. 117 shall indicate that quark- and charged lepton flavor- entries – tricolored for quarks and colorless for charged leptons – are arranged sequentially . Taking e.g. up and down quark flavors and  $e^-$ ,  $\mu^-$  charged lepton flavors the transposed row vector  $f_\bullet^T$

$$f^T = \{ u_r, u_g, u_b ; d_r, d_g, d_b ; e^-, \mu^- \} \quad (118)$$

Generalizations beyond the six known color triplet quark flavors  $u, d ; c, s ; t, b$  and the three equally charged leptons  $e^-, \mu^-, \tau^-$  to colored flavors in other irreducible representations of  $SU3_c$ , as well as to charged leptons with charges different from  $e^-, \mu^-, \tau^-$  are then straightforward , whereby the spins shall be restricted to  $\frac{1}{2}$  ( fermions ) and the union of color representations is limited such as not to upset asymptotic freedom in the ultraviolet for  $SU3_c$ .<sup>2</sup>

This said, the matrices  $d_r, d_{em}$  in eq. 117 over the union of irreducible color representations  $\bigcup \mathcal{D}$  is suitably extended , are block diagonal for  $d_r(\bigcup \mathcal{D})$  and  $d_{em}(\bigcup \ell)$ , with

$$[d_r, d_{em}] = 0 ; r = 1, \dots, 8 \quad (119)$$

For the flavor set in eq. 118 the 9 matrices (  $8 \times 8$  )  $d_r, d_{em}$  are

$$d_r = \frac{1}{i} \begin{bmatrix} \frac{1}{2} \lambda_r & 0 & 0 \\ 0 & \frac{1}{2} \lambda_r & 0 \\ 0 & 0 & 0 \end{bmatrix} ; d_{em} = \frac{1}{i} \begin{bmatrix} \frac{2}{3} \mathbb{1}_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 2} \\ 0_{3 \times 3} & -\frac{1}{3} \mathbb{1}_{3 \times 3} & 0_{3 \times 2} \\ 0_{2 \times 3} & 0_{2 \times 3} & -\mathbb{1}_{2 \times 2} \end{bmatrix} \quad (120)$$

<sup>2</sup> The ultraviolet stability of QCD , whence isolated from QED is obviously not sufficient to establish its asymptotic ultraviolet behavior in the context of QCD-QED, because of the opposite stability - instability behaviour of QCD relative to QED, considered in isolation one from the other. In the present subsection we are not concerned with the full QCD-QED complex .

The sub-block sizes are indicated in the entries for  $d_{em}$  in eq. 120 . We retain the form of the electromagnetic potentials ( eq. 117 )

$$\begin{aligned} \mathcal{W}_\mu^{em} &= W_\mu^{em} d_{em} ; d_{em} = \frac{1}{i} Q_{em} \\ Q_{em} &= \text{diag} ( Q_{f_1} , \dots Q_{f_N} ) ; Q_{f_\bullet} = e ( f_\bullet ) / e \end{aligned} \quad (121)$$

The diagonal elements of the ( hermitian ) matrix  $Q_{em}$  as defined in eq. 121 are the relative electric charges of the individual fermions to the elementary charge , of the proton say , identical for all members transforming under an irreducible representation of  $SU3_c$  .

In a first step we extend the Lagrangean density  $\mathcal{L}_{\{q\}}$  in eq. 36 for color triplet quark flavors, to which we restrict the  $f_\bullet$  components considered here, to QCD-QED

$$\begin{aligned} \mathcal{L}_{\{q\}} &= \sum_{q-fl} \bar{q}^{\dot{c}'} \left\{ \frac{i}{2} \gamma^\mu \left[ \begin{pmatrix} \overrightarrow{D}_\mu (3) \end{pmatrix}_{c'\dot{c}} - \begin{pmatrix} \overleftarrow{D}_\mu (\bar{3}) \end{pmatrix}_{\dot{c}c'} \right] - m_q \delta_{c'\dot{c}} \right\} q^c \\ \begin{pmatrix} \overrightarrow{D}_\mu (3) \end{pmatrix}_{c'\dot{c}} &= \overrightarrow{\partial}_\mu \delta_{c'\dot{c}} + W_\mu^r \frac{1}{i} \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}} + \\ &\quad + W_\mu^{em} \frac{1}{i} Q_q \delta_{c'\dot{c}} \\ \begin{pmatrix} \overleftarrow{D}_\mu (\bar{3}) \end{pmatrix}_{\dot{c}c'} &= \overleftarrow{\partial}_\mu \delta_{\dot{c}c'} - W_\mu^r \frac{1}{i} \left( \frac{1}{2} \bar{\lambda}^r \right)_{\dot{c}c'} - \\ &\quad - W_\mu^{em} \frac{1}{i} Q_q \delta_{\dot{c}c'} \\ &= \overleftarrow{\partial}_\mu \delta_{c'\dot{c}} - W_\mu^r \frac{1}{i} \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}} - \\ &\quad - W_\mu^{em} \frac{1}{i} Q_q \delta_{c'\dot{c}} \end{aligned} \quad (122)$$

Eq. 37 becomes extending to all fermion flavors  $f = ( q , \ell^\alpha )$

$$\begin{aligned} \mathcal{L}_{\{q\}} &= \sum_{q-fl} \bar{q}^{\dot{c}'} \left\{ \gamma^\mu \left[ \begin{aligned} &\frac{i}{2} \overrightarrow{\partial}_\mu \delta_{c'\dot{c}} \\ &+ W_\mu^r \left( \frac{1}{2} \lambda^r \right)_{c'\dot{c}} \\ &+ W_\mu^{em} Q_q \delta_{c'\dot{c}} \end{aligned} \right] - m_q \delta_{c'\dot{c}} \right\} q^c \\ \mathcal{L}_{\{\ell\}} &= \sum_{\ell-fl} \bar{\ell}^{\dot{\alpha}'} \left\{ \gamma^\mu \left[ \begin{aligned} &\frac{i}{2} \overrightarrow{\partial}_\mu \delta_{\alpha'\dot{\alpha}} \\ &+ W_\mu^{em} Q_{\ell_\alpha} \delta_{\alpha'\dot{\alpha}} \end{aligned} \right] - m_{\ell_\alpha} \delta_{\alpha'\dot{\alpha}} \right\} \ell^\alpha \\ \mathcal{L}_{\{f\}} &= \mathcal{L}_{\{q\}} + \mathcal{L}_{\{\ell\}} \end{aligned} \quad (123)$$

For the photon potentials (connection) and fieldstrengths we perform the analogous steps as for the nonabelian counterparts, repeating for the latter

eq. 25

$$\begin{aligned}
\mathcal{W}^{(2)}(\mathcal{D}) &\rightarrow \mathcal{W}^{(2)} = \partial \mathcal{W}^{(1)} + (\mathcal{W}^{(1)})^2 ; \partial \equiv dx^\mu \partial_{x^\mu} \\
(\mathcal{W}^{(2)})_{\alpha\beta} &= \frac{1}{2} W_{\mu\nu}^r (d_r)_{\alpha\beta} dx^\mu \wedge dx^\nu \\
d_r &\in Lie(\mathcal{D}) \rightarrow \\
\mathcal{W}_{\mu\nu}^{(2)} &= \partial_\mu \mathcal{W}_\nu^{(1)} - \partial_\nu \mathcal{W}_\mu^{(1)} + [\mathcal{W}_\mu^{(1)}, \mathcal{W}_\nu^{(1)}] \\
W_{\mu\nu}^r &= -W_{\nu\mu}^r = \partial_\mu W_\nu^r - \partial_\nu W_\mu^r + f_{rpq} W_\mu^p W_\nu^q \\
\hline
\mathcal{W}^{(2)}(\mathcal{D}) &\equiv \mathcal{B}^{(2)}(\mathcal{D}) ; W_{\mu\nu}^r \equiv B_{\mu\nu}^r \quad \begin{array}{l} \text{components of field} \\ \text{strengths independent} \\ \text{of } \mathcal{D} \end{array}
\end{aligned} \tag{124}$$

The electromagnetic counterparts are ( eqs. 122 , 123 )

$$\begin{aligned}
\mathcal{W}_\mu^{em} &= W_\mu^{em} d_{em}(f) \\
(d_{em})_{\alpha'\alpha} &= \frac{1}{i} Q_{f\alpha} \delta_{\alpha'\alpha} ; [d_{em}, d_r(\cup \mathcal{D}(f))] = 0 \\
W_{\mu\nu}^{em} &\equiv B_{\mu\nu}^{em} = \partial_\mu W_\nu^{em} - \partial_\nu W_\mu^{em} \\
\hline
L^{em} &= -\frac{1}{4e^2} B_{\mu\nu}^{em} B^{\mu\nu em} + \mathcal{L}_{\{f\}}
\end{aligned} \tag{125}$$

The electromagnetic Euler-Lagrange equations extending eqs. 33, 34, 39, 107 and 116 become

$$\begin{aligned}
\partial_\varrho \left\{ \frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^{em}} \frac{\delta B_{\mu\nu}^{em}}{\delta (\partial_\varrho W_\sigma^{em})} \right\} &= \frac{\delta \mathcal{L}_{\{f\}}}{\delta W_\sigma^{em}} = j^{\sigma em} \\
\frac{\delta \mathcal{L}}{\delta B_{\mu\nu}^{em}} &= \frac{1}{2e^2} B^{\nu\mu em} ; \frac{\delta B_{\mu\nu}^{em}}{\delta (\partial_\varrho W_\sigma^{em})} = \delta_\mu^\varrho \delta_\nu^\sigma - \delta_\nu^\varrho \delta_\mu^\sigma
\end{aligned} \tag{126}$$

The electromagnetic Euler-Lagrange equations, i.e. the inhomogeneous Maxwell equations, thus take the form

$$\begin{aligned}
\partial_\varrho E B^{\sigma\varrho em} &= j^{\sigma em} = \sum_\alpha \bar{f}_\alpha \gamma^\sigma Q_{f\alpha} f_\alpha \\
E &= 1/e^2
\end{aligned} \tag{127}$$

We are now in the position to make contact with the  $dim [M^4]$  equations for QCD – eq. 108 repeated below

$$\begin{aligned}
G &= g^{-2} \\
D_\tau D_\varrho \{ G B_\sigma^\varrho \} - D_\sigma D_\varrho \{ G B_\tau^\varrho \} &= D_\tau j_\sigma - D_\sigma j_\tau
\end{aligned} \tag{128}$$

The QED analogous equations are derived from the equations of motion ( eq. 127 )

$$\partial^\varrho \left( \partial_\tau B_{\sigma\varrho}^{em} - \partial_\sigma B_{\tau\varrho}^{em} \right) = e^2 \left( \partial_\tau j_\sigma^{em} - \partial_\sigma j_\tau^{em} \right) \quad (129)$$

The essential difference between the nonabelian chromo-dipole equations ( eq. 128 ) and their  $U1^{em}$  counterparts ( eq. 129 ) is that the former are  $SU3_c$  covariant, both sides of eq. 128 transforming according to the adjoint- ( i.e. octet- ) representation of the local  $SU3_c$  gauge group, whereas the electromagnetic counterpart as displayed in eq. 129 is gauge invariant under the local  $U1^{em}$  gauge group, and thus independent of the electromagnetic potentials  $W_\mu^{em} \equiv -w_\mu^{em}$ .

Nevertheless in both cases the homogeneous Bianchi identities for ( covariant ) derivatives of the field strength are used as analog identities to the case of QCD in eq. 114, repeated below

$$\begin{aligned} D_\tau B_{\sigma\varrho} - D_\sigma B_{\tau\varrho} &= D_\tau B_{\sigma\varrho} + D_\sigma B_{\varrho\tau} = -D_\varrho B_{\tau\sigma} \longrightarrow \\ \text{Bianchi identity : } D_\tau B_{\sigma\varrho} + D_\sigma B_{\varrho\tau} + D_\varrho B_{\tau\sigma} &= 0 \end{aligned} \quad (130)$$

and the QED ( abelian ) analog Bianchi identity , i.e. the homogeneous Maxwell equations

$$\begin{aligned} \partial_\tau B_{\sigma\varrho}^{em} - \partial_\sigma B_{\tau\varrho}^{em} &= \partial_\tau B_{\sigma\varrho}^{em} + \partial_\sigma B_{\varrho\tau}^{em} = -\partial_\varrho B_{\tau\sigma}^{em} \longrightarrow \\ \text{Bianchi identity : } \partial_\tau B_{\sigma\varrho}^{em} + \partial_\sigma B_{\varrho\tau}^{em} + \partial_\varrho B_{\tau\sigma}^{em} &= 0 \end{aligned} \quad (131)$$

As in the pair of eqations 128 (  $SU3_c$  ) and 129 (  $U1^{em}$  ) , eq. 130 is gauge covariant with respect to the local  $SU3_c$  gauge group, whereas eq. 131 is gauge invariant with respect to the local  $U1^{em}$  gauge group . Substituting eq. 131 in eq. 129 we obtain the electromagnetic dipole equations for the electrodynamic field strengths

$$\begin{aligned} \square B_{\sigma\tau}^{em} &= e^2 \left( \partial_\tau j_\sigma^{em} - \partial_\sigma j_\tau^{em} \right) \\ \square &= \partial^\varrho \partial_\varrho \end{aligned} \quad (132)$$

Eq. 132 is readily compared with its nonabelian analog , eq. 115 , repeated below

$$\left[ \begin{aligned} &D^\varrho D_\varrho B_{\sigma\tau} + \\ &+ \mathcal{B}_{\tau\varrho} B_{\sigma}^\varrho - \mathcal{B}_{\sigma\varrho} B_{\tau}^\varrho \end{aligned} \right]^s = g^2 \left( D_\tau j_\sigma - D_\sigma j_\tau \right)^s ;$$


---

$s = 1, \dots, 8$  : color octet label

(133)

We conclude this section repeating the definitions in eq. 35 relative to the adjoint covariant derivative and the field strength components pertaining to

the adjoint representation , appearing in shorthand notation in eqs. 115 and 133

$$\begin{aligned}
(D_{\varrho}(ad))_{sr} &= \partial_{\varrho} \delta_{sr} + W_{\varrho}^t(ad_t)_{sr} \\
B_{\sigma\tau} &= (B_{\sigma\tau})^t = B_{\sigma\tau}^t \\
(\mathcal{B}_{\sigma\tau})_{sr} &= B_{\sigma\tau}^t(ad_t)_{sr} ; (ad_t)_{sr} = f_{str}
\end{aligned} \tag{134}$$

The chromo-dipole equation ( eqs. 133 - 134 ) was derived – but considering the Lagrangean  $\mathcal{L} \sim \frac{1}{4} B_{\mu\nu}^s B^{\mu\nu s}$  *not*  $\overline{\mathcal{L}}$  as given in eq. 78 – only recently by the author of these notes. To my knowledge it represents a new element.

## References

- [1-2011] P. Minkowski, 'QCD and the incomplete links between mass and gauge', massgauge-kyoto2011.pdf,  
(URL) <http://www.mink.itp.unibe.ch/lectures.html> ,  
'The phase structure of QCD and its eventual root in gauge boson pair-condensation', chapter 3 on complete connections, notefile, unpublished.
- [2-1968] G. Segal, 'The representation ring of a compact Lie group', Publications Mathématiques de L'IHS, Volume 34, Number 1 (1968) 113-128, DOI: 10.1007/BF02684592 , and references cited therein .
- [3-1981] P. Minkowski, 'On The Ground State Expectation Value Of The Field Strength Bilinear In Gauge Theories And Constant Classical Fields', Nuclear Physics B 177 (1981) 203-217 , and references cited therein .
- [4-1978] Heinz Pagels and E. Tomboulis, 'Vacuum of the quantum Yang-Mills theory and magnetostatics', Nuclear Physics B143 (1978) 485-502.
- [5-1918] E. Nöther, "Invariante Variationsprobleme",  
Nachrichten der Königlich Gesellschaft der Wissenschaften zu  
Göttingen, mathematisch-physikalische Klasse 1918 , 235-257 ;  
english translation :  
Emmy Noether, M. A. Tavel, 'Invariant Variation Problems', (Submitted on 8 Mar 2005), arXiv:physics/0503066v1 [physics.hist-ph] .
- [6-1940] F. J. Belinfante, 'On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields', Physica 7 (1940) 449-474 , a full derivation can be found in the textbook by Res Jost [9-1965] .

- [7-1976] P. Minkowski, 'On the anomalous divergence of the dilatation current in gauge theories', Bern preprint 1976 , unpublished , URL : <http://www.mink.itp.unibe.ch/publications.html> [ file : all.pdf ] .
- For fairness of citation let me quote the papers published in 1977 on the trace anomaly
- S. L. Adler, J. C. Collins and A. Duncan, Phys. Rev. D15 (1977) 1712,  
N. K. Nielsen. Nucl. Phys. B 120 (1977) 212,  
S. L. Adler, A. Duncan and S. D. Joglekar, Phys. Rev. D 16 (1977) 438.
- [8-1975] H. Kluberg-Stern and J. B. Zuber, 'Ward Identities and Some Clues to the Renormalization of Gauge Invariant Operators', Phys.Rev. D12 (1975) 467 ,  
'Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions', Phys.Rev. D12 (1975) 482 and  
'Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 2. Gauge Invariant Operators', Phys.Rev.D12 (1975) 3159-3180.
- [9-1965] R. Jost, 'The general theory of quantized fields', American Mathematical Society, Providence, Rhode Island , 1965 .
- [10-1988] S. G. Gorishnii , A. L. Kataev and S. A. Larin, 'Next-To-Leading  $O(\alpha_s^3)$  QCD Correction to  $\Sigma_t (e^+ e^- \rightarrow \text{Hadrons})$  : Analytical Calculation and Estimation of the Parameter Lambda (MS)', JINR-E2-88-254, Apr 1988. 18pp., Phys.Lett.B212 (1988) 238-244 .
- [11-1997] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, 'The Four loop beta function in quantum chromodynamics', UM-TH-97-01, NIKHEF-97-001, Jan 1997. 9pp.,  
Phys.Lett.B400 (1997) 379 , hep-ph/9701390 .
- [12-1979] M. A. Shifman , A. I. Vainshtein and V. I. Zakharov , 'QCD and resonance physics. Theoretical Foundations.', Nucl. Phys. B147 (1979) 385 , 448 .
- [13-2011] S. Narison, 'Gluon Condensates and  $m_{c,b}$  from QCD-Moments and their ratios to Order  $\alpha_s^3$  and  $\langle G^4 \rangle$ ', May 2011. 10pp., Phys.Lett.B706 (2012) 412-422 , arXiv:1105.2922 [hep-ph] .
- [14-1975] C. Becchi, A. Rouet and R. Stora, Comm. Math. Phys. 42 (1975) 127 ,  
Ann. Phys. 98 (1976) 287 , and in 'Renormalization Theory', G. Velo and A. S. Wightman eds.,  
Reidel , Dordrecht 1976 .
- [15-1975] I. V. Tyutin, Lebedev Institute preprint N39 , 1975 .

- [16-1972] H. Fritzsch and M. Gell-Mann, 'Current algebra: Quarks and what else?', Proceedings of 16th International Conference on High-Energy Physics, Batavia, Illinois, 6-13 Sep 1972, published in eConf C720906V2 (1972) 135-165, also in Physics, Proceedings of the XVI International Conference on High Chicago 1972 p.135 (J. D. Jackson, A. Roberts, eds.), hep-ph/0208010 .
- [17-1973] H. Fritzsch, Murray Gell-Mann and H. Leutwyler , 'Advantages of the Color Octet Gluon Picture', CALT-68-409, 1973, Phys.Lett.B47 (1973) 365-368 .
- [18-1973(4)] D.J. Gross and F. Wilczek,  
     'Asymptotically free gauge theories. 1',Phys.Rev.D8 (1973) 3633,  
     'Asymptotically free gauge theories. 2',Phys.Rev.D9 (1974) 980.
- [19-1973(4)] D. Politzer, ' Reliable Perturbative Results for Strong Interactions?', Phys. Rev. Lett. 30 (1973) 1346 ,  
     H. Georgi and D. Politzer, 'Electroproduction scaling in an asymptotically free theory of strong interactions', Phys. Rev. D9 (1974) 416 .